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Advancing Statistical Model Flexibility through the Development of the Exponentiated Generalized Topp Leone-Fréchet Distribution

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ABSTRACT

We propose a new flexible statistical model called the Exponentiated Generalized Topp Leone-Fréchet (EGTL-F) distribution, obtained by compounding the Exponentiated Generalized Topp Leone-G (EGTL-G) family with the heavy-tailed Fréchet distribution. The EGTL-F distribution combines the skewness-control properties of the Topp Leone transformation with the extreme-value tail behaviour of the Fréchet distribution, making it suitable for modelling positively skewed data with heavy tails. We derive its cumulative distribution function, probability density function, survival function, hazard function, quantile function, ordinary moments, order statistics, and Rényi entropy. Parameters are estimated using the method of Maximum Likelihood Estimation (MLE). A Monte Carlo simulation study shows that the bias and mean squared error (MSE) of the MLEs decrease as the sample size increases from $n = 10$ to $n = 100$, confirming the consistency of the estimators. Two real datasets, the strength of 1.5 cm glass fibres and the relief times following an analgesic treatment, are used to compare the EGTL-F distribution against three existing competitor distributions: the Exponentiated Fréchet distribution (EFD), the Fréchet distribution (FD), and the Exponential distribution (ED). In both applications, the EGTL-F distribution achieves the highest log-likelihood and the lowest Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values, indicating a superior fit. These results show that the EGTL-F distribution is a competitive and flexible model for right-skewed, heavy-tailed lifetime data.

1. INTRODUCTION

The Fréchet distribution is a well-known heavy-tailed distribution that is appropriate for modelling extreme values, and it is applied in disciplines such as environmental science, hydrology, finance, and engineering, where extreme values are of primary interest (Gómez et al., 2024; Lee & Scott, 2022). However, in many of these fields, classical two-parameter distributions are often too rigid to capture data behaviour that combines strong asymmetry with unusually long tails. This limitation has motivated continuous extensions of classical distribution theory, particularly through Exponentiated

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Generalized families, since distributions with more than two parameters allow extra control over skewness and kurtosis (Almalki & Nadarajah, 2014; Rannona & Simbarashe, 2022).

Building on this line of work, the present paper compounds the EGTL-G family with the Fréchet distribution to obtain the Exponentiated Generalized Topp Leone-Fréchet (EGTL-F) distribution. The EGTL-F distribution takes the skewness-control merit of the Topp Leone transformation and combines it with the Fréchet distribution's recognised strength in modelling extreme values, producing a five-parameter model intended for asymmetric, heavy-tailed data.

The objectives of this paper are: (i) to derive the EGTL-F distribution and its key mathematical properties, including the quantile function, ordinary moments, order statistics, and Rényi entropy; (ii) to estimate its parameters using maximum likelihood and assess estimator performance through a Monte Carlo simulation study; and (iii) to demonstrate the practical usefulness of the EGTL-F distribution by fitting it to two real datasets and comparing its performance against existing competitor distributions using standard model-selection criteria.

2. LITERATURE REVIEW

One useful route for building such flexible families is the Topp Leone-G family of distributions, introduced by Al-Shomrani et al. (2016) and later generalized by Reyad et al. (2019) into the Exponentiated Generalized Topp Leone-G (EGTL-G) family. The Topp Leone transformation is well suited to controlling skewness, while the Fréchet distribution is well suited to framing upper-tail (extreme-value) behaviour (de Haan & Ferreira, 2006). Other recent extensions in this same spirit, combining a flexible generator family with a heavy-tailed baseline distribution, have produced distributions with improved tail and skewness properties for lifetime and reliability data (Alzahrani & Sagor, 2020; Heysham et al., 2019; da Silva et al., 2013; Teamah et al., 2020a; Teamah et al., 2020b).

3. METHODOLOGY

This section presents the derivation of the proposed EGTL-F distribution and the statistical tools used to study its properties.

3.1 The Exponentiated Generalized Topp Leone-G Family

The Exponentiated Generalized Topp Leone-G (EGTL-G) family of distributions was introduced by Reyad et al. (2019), building on the earlier Topp Leone-G family of Al-Shomrani et al. (2016). For a baseline distribution with cumulative distribution function (CDF) $G(x)$, the CDF and probability density function (PDF) of the EGTL-G family are given, respectively, by

$$F_{EGTL\text{-}G}(x) = 1-(1-(1-[1-G(x)^2]^{\lambda})^{a})^{b}, \qquad a,b,\lambda>0, \tag{1}$$

$$f_{EGTL\text{-}G}(x) = 2ab\lambda\, g(x)\, G(x)[1-G(x)^2]^{\lambda-1}(1-[1-G(x)^2]^{\lambda})^{a-1}(1-(1-[1-G(x)^2]^{\lambda})^{a})^{b-1}, \tag{2}$$

where $a,b,\lambda>0$ are shape parameters that govern skewness and tail weight, and $g(x)$ is the baseline PDF corresponding to $G(x)$.

The Fréchet distribution, which is used in this paper as the baseline distribution, has CDF, PDF, and quantile function given by

$$F_F(x) = e^{-(\theta/x)^{\beta}} \qquad x>0\ \beta\ and\ \theta>0 \tag{3}$$

$$f_F(x) = \beta\theta^{\beta}x^{-(\beta+1)}e^{-(\theta/x)^{\beta}}, \quad x>0\ \beta\ and\ \theta>0 \tag{4}$$

$$Q_F(p) = \theta(-\ln p)^{-1/\beta}, \tag{5}$$

where $\theta>0$ is a scale parameter and $\beta>0$ is a shape parameter.

3.2 The Proposed Exponentiated Generalized Topp Leone-Fréchet (EGTL-F) Distribution

The EGTL-F distribution is obtained by substituting the Fréchet baseline into the EGTL-G family. Throughout this paper, we write $G(x) = e^{-(\theta/x)^{\beta}}$ for the Fréchet CDF, so that $1-G(x)^2 = 1-e^{-2(\theta/x)^{\beta}}$.

Cumulative Distribution Function (CDF). The CDF of the EGTL-F distribution is

$$F_{EGTL\text{-}F}(x) = 1 - \left(1 - \left(1 - \left[1 - e^{-2\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\lambda}\right)^{a}\right)^{b}, \qquad x, \beta, \lambda, a, b, \theta > 0. \tag{6}$$

Probability Density Function (PDF). Differentiating Eq. (6) with respect to x gives the PDF of the EGTL-F distribution as

$$f_{EGTL\text{-}F}(x) = 2ab\lambda\beta\theta^{\beta}x^{-(\beta+1)}e^{-2(\theta/x)^{\beta}}\left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda-1}\left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a-1}$$
$$\times \left(1 - \left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a}\right)^{b-1}, \quad x, \beta, \lambda, a, b, \theta > 0. \tag{7}$$

Quantile Function. The quantile function $Q(p)$ is obtained by inverting the CDF in Eq. (6), that is, by solving $F_{EGTL\text{-}F}(x) = p$ for x. Setting $G(x) = e^{-2(\theta/x)^{\beta}}$ and solving step by step,

$$1 - p = \left(1 - (1 - (1 - G)^{\lambda})^{a}\right)^{b} \;\Rightarrow\; (1 - p)^{1/b} = 1 - (1 - (1 - G)^{\lambda})^{a}$$
$$\Rightarrow\; (1 - (1 - G)^{\lambda})^{a} = 1 - (1 - p)^{1/b} \;\Rightarrow\; 1 - (1 - G)^{\lambda} = \left[1 - (1 - p)^{1/b}\right]^{1/a}$$
$$\Rightarrow\; (1 - G)^{\lambda} = 1 - \left[1 - (1 - p)^{1/b}\right]^{1/a} \;\Rightarrow\; G = 1 - \left(1 - \left[1 - (1 - p)^{1/b}\right]^{1/a}\right)^{1/\lambda},$$

so that, since $G(x) = e^{-2(\theta/x)^{\beta}}$, the quantile function is

$$Q(p) = \theta\left[-\frac{1}{2}\ln\left(1 - \left[1 - \left(1 - p^{1/b}\right)^{1/a}\right]^{1/\lambda}\right)\right]^{-1/\beta}, \qquad 0 < p < 1. \tag{8}$$

Median. Setting $p = 0.5$ in Eq. (8) gives the median of the EGTL-F distribution as

$$Q(0.5) = \theta\left[-\frac{1}{2}\ln\left(1 - \left[1 - \left(1 - 0.5^{1/b}\right)^{1/a}\right]^{1/\lambda}\right)\right]^{-1/\beta}. \tag{9}$$

Survival Function. The survival function of the EGTL-F distribution is

$$S_{EGTL\text{-}F}(x) = 1 - F_{EGTL\text{-}F}(x) = 1 - \left(1 - \left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a}\right)^{b}, \qquad x, \beta, \lambda, a, b, \theta > 0.$$

Hazard Function. The hazard function is defined as $h_{EGTL\text{-}F}(x) = f_{EGTL\text{-}F}(x)/S_{EGTL\text{-}F}(x)$, which gives

$$h_{EGTL\text{-}F}(x) = \frac{2ab\lambda\beta\theta^{\beta}x^{-(\beta+1)}e^{-2(\theta/x)^{\beta}}\left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda-1}\left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a-1}}{\left(1 - \left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a}\right)^{b}}$$
$$\times \left(1 - \left(1 - \left[1 - e^{-2(\theta/x)^{\beta}}\right]^{\lambda}\right)^{a}\right)^{b-1} \quad x, \beta, \lambda, a, b, \theta > 0. \tag{10}$$

3.3 Binomial Series Expansion of the PDF

To facilitate the derivation of moments and other properties, we use the binomial series expansion

$$(1 - e^{-z})^{\alpha} = \sum_{i=0}^{\infty}\binom{\alpha}{i}(-1)^{i}e^{-zi}. \tag{11}$$

Applying this expansion successively to the three bracketed terms in Eq. (7), the PDF of the EGTL-F distribution can be written as the linear (series) representation

$$f_{EGTL\text{-}F}(x) = 2ab\beta\lambda\theta^{\beta}x^{-(\beta+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{b-1}{i}\binom{a+ai-1}{j}\binom{\lambda+\lambda j-1}{k}(-1)^{i+j+k}e^{-2(k+1)(\theta/x)^{\beta}}. \quad (12)$$

Writing

$$w_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{b-1}{i}\binom{a+ai-1}{j}\binom{\lambda+\lambda j-1}{k}(-1)^{i+j+k}, \quad (13)$$

the series representation in Eq. (13) simplifies to Eq. (14).

$$f_{EGTL\text{-}F}(x) = 2ab\beta\lambda\theta^{\beta}\sum_{k=0}^{\infty}w_k\, x^{-(\beta+1)}e^{-2(k+1)(\theta/x)^{\beta}}. \quad (14)$$

3.4 Ordinary Moments

The r-th ordinary moment of the EGTL-F distribution is given by Eq. (15) and (16).

$$E(X^r) = \int_0^{\infty} x^r\, f_{EGTL\text{-}F}(x)\, dx \quad (15)$$

$$E(X^r) = \int_0^{\infty} x^{r-\beta-1}\, 2ab\beta\lambda\theta^{\beta}\sum_{k=0}^{\infty}w_k\, e^{-2(k+1)(\theta/x)^{\beta}}\, dx. \quad (16)$$

Using the substitution $u = 2(k+1)(\theta/x)^{\beta}$, so that $x = \theta[2(k+1)/u]^{1/\beta}$ and

$$dx = -\frac{\theta}{\beta}[2(k+1)]^{\frac{1}{\beta}}u^{-\frac{1}{\beta}-1}\, du,$$

The integral in Eq. (16) reduces to a standard Gamma-function form, giving in Eq. (17).

$$E(X^r) = ab\lambda\,\theta^r\, 2^{r/\beta}\sum_{k=0}^{\infty}w_k\,(k+1)^{r/\beta-1}\,\Gamma\left(1-\frac{r}{\beta}\right), \qquad r < \beta. \quad (17)$$

Existence condition. Because the EGTL-F distribution inherits the Fréchet baseline's polynomial upper-tail decay, the Gamma function $\Gamma(1-r/\beta)$ is finite only when its argument is positive. The r-th ordinary moment of the EGTL-F distribution therefore **exists only when** $r < \beta$; for $r \geq \beta$, the corresponding moment is infinite. In particular, the mean exists only if $\beta > 1$, the variance only if $\beta > 2$, and the third and fourth moments (needed for skewness and kurtosis) require $\beta > 3$ and $\beta > 4$, respectively. This constraint must be checked against the fitted value of β before reporting any moment-based summary of the distribution.

Setting $r = 1,2,3,4$ in Eq. (17) gives the first four moments as shown in Eq. (18) to Eq. (21).

$$\mu = E(X) = ab\lambda\theta\, 2^{1/\beta}\sum_{k=0}^{\infty}w_k\,(k+1)^{1/\beta-1}\Gamma\left(1-\frac{1}{\beta}\right), \qquad \beta > 1, \quad (18)$$

$$\mu_2' = E(X^2) = ab\lambda\theta^2 2^{\frac{2}{\beta}}\sum_{k=0}^{\infty}w_k\,(k+1)^{\frac{2}{\beta}-1}\Gamma\left(1-\frac{2}{\beta}\right), \qquad \beta > 2, \quad (19)$$

$$\mu_3' = E(X^3) = ab\lambda\theta^3 2^{\frac{3}{\beta}}\sum_{k=0}^{\infty}w_k\,(k+1)^{\frac{3}{\beta}-1}\Gamma\left(1-\frac{3}{\beta}\right), \qquad \beta > 3, \quad (20)$$

$$\mu_4' = E(X^4) = ab\lambda\theta^4 2^{\frac{4}{\beta}}\sum_{k=0}^{\infty}w_k\,(k+1)^{\frac{4}{\beta}-1}\Gamma\left(1-\frac{4}{\beta}\right), \qquad \beta > 4, \quad (21)$$

From which the skewness and kurtosis follow in the usual way, this is presented in Eq. (22) and (23).

$$SK = \frac{\mu_3{}' - 3\mu_2{}'\mu + 2\mu^3}{(\mu_2{}' - \mu^2)^{\frac{3}{2}}} \quad (22)$$

$$KS = \frac{\mu_4{}' - 4\mu_3{}'\mu + 6\mu_2{}'\mu^2 - 3\mu^4}{(\mu_2{}' - \mu^2)^2} \tag{23}$$

Substitute Eq. (18) – (21) into Eq. (22) and (23)

3.5 Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be an observed random sample from the EGTL-F distribution. The likelihood function for $\eta = (a, b, \beta, \lambda, \theta)$ is

$$L(\eta) = \prod_{i=1}^{n} f_{EGTL\text{-}F}\,(x_i) \tag{24}$$

And the log-likelihood function, obtained by substituting Eq. (7) for each x_i and taking logarithms, is

$$\begin{aligned}\ell(\eta) &= n\ln 2 + n\ln a + n\ln b + n\ln\lambda + n\ln\beta + n\beta\ln\theta - (\beta+1)\sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta}\\ &+(\lambda-1)\sum_{i=1}^{n}\ln\left[1 - e^{-2(\theta/x_i)^{\beta}}\right] + (a-1)\sum_{i=1}^{n}\ln\left[1 - \left(1 - e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda}\right]\\ &+(b-1)\sum_{i=1}^{n}\ln\left[1 - \left(1 - \left(1 - e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda}\right)^{a}\right].\end{aligned} \tag{25}$$

Differentiating Eq. (25) with respect to each parameter gives the score Eqs. (26) – (29):

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= \frac{n}{a} + \sum_{i=1}^{n}\ln\left[1 - \left(1 - e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda}\right]\\ &-(b-1)\sum_{i=1}^{n}\frac{\left[1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a}\ln\left[1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]}{1 - \left[1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a}}\end{aligned} \tag{26}$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln\left[1 - \left(1 - \left(1 - e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda}\right)^{a}\right] \tag{27}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n}\ln\left[1 - e^{-2(\theta/x_i)^{\beta}}\right] - (a-1)\sum_{i=1}^{n}\frac{\left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\ln\left[1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right]}{1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}}\\ &-(b-1)a\sum_{i=1}^{n}\frac{\left[1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a-1}\left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\ln\left[1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right]}{1 - \left[1 - \left(1 - e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a}}\end{aligned} \tag{28}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + n\ln\theta - \sum_{i=1}^{n} \ln x_i - 2\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\beta} \ln\left(\frac{\theta}{x_i}\right) + 2(\lambda-1)\sum_{i=1}^{n} \frac{(\theta/x_i)^{\beta} e^{-2(\theta/x_i)^{\beta}} \ln(\theta/x_i)}{1-e^{-2(\theta/x_i)^{\beta}}}$$
$$-2\lambda(a-1)\sum_{i=1}^{n} \frac{(\theta/x_i)^{\beta} e^{-2(\theta/x_i)^{\beta}} \left(1-e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda-1} \ln(\theta/x_i)}{1-\left(1-e^{-2(\theta/x_i)^{\beta}}\right)^{\lambda}}$$
$$-2\lambda a(b-1)\sum_{i=1}^{n} \frac{\left(\frac{\theta}{x_i}\right)^{\beta} e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}} \left(1-e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda-1} \left[1-\left(1-e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a-1} \ln\left(\frac{\theta}{x_i}\right)}{1-\left[1-\left(1-e^{-2\left(\frac{\theta}{x_i}\right)^{\beta}}\right)^{\lambda}\right]^{a}} \quad (29)$$

The MLEs $\hat{a}, \hat{b}, \hat{\beta}, \hat{\lambda}, \hat{\theta}$ are obtained by setting Eqs. (26) – (29), together with the corresponding Eq. for $\partial\ell/\partial\theta$, equal to zero and solving the resulting system numerically, since no closed-form solution exists.

3.6 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from the EGTL-F distribution, and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ denote the corresponding order statistics. The probability density function of the i-th order statistic, $X_{(i)}$, for $i = 1,2,\ldots,n$, is given by the standard formula in Eq. (29).

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x)\,[F(x)]^{i-1}\,[1-F(x)]^{n-i}, \qquad i = 1,\ldots,n, \quad (30)$$

where $B(i, n-i+1) = (i-1)!\,(n-i)!/n!$ is the Beta function.

Substituting the EGTL-F CDF (Eq. 6) and PDF (Eq. 7) into Eq. (29) gives

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f_{EGTL\text{-}F}(x)[F_{EGTL\text{-}F}(x)]^{i-1}[1-F_{EGTL\text{-}F}(x)]^{n-i} \quad (31)$$

Applying the binomial series expansion of Eq. (12) to the term $[1-F_{EGTL\text{-}F}(x)]^{n-i}$, and using the series form of $f_{EGTL\text{-}F}(x)$ from Eq. (15), the density of the i-th order statistic can, after simplification, be written as the double series

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} 2ab\beta\lambda\theta^{\beta} \sum_{k=0}^{\infty}\sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m w_k\, x^{-(\beta+1)} [F_{EGTL\text{-}F}(x)]^{i-1+m} e^{-2(k+1)(\theta/x)^{\beta}} \quad (32)$$

3.7 Rényi Entropy

The Rényi entropy of a random variable X with density $f(x)$ is defined, for $\rho > 0,\ \rho \ne 1$, as

$$I_R(\rho) = \frac{1}{1-\rho} \log\left(\int_0^{\infty} [f(x)]^{\rho}\, dx\right) \quad (33)$$

To evaluate Eq. (32) for the EGTL-F distribution, we raise the PDF in Eq. (7) to the power ρ and re-expand it using the binomial series of Eq. (12), applied successively in the same way as in Section 2.3. Writing $G = e^{-2(\theta/x)^{\beta}}$ for brevity, this gives

$$[f_{EGTL\text{-}F}(x)]^{\rho} = \left(2ab\beta\lambda\theta^{\beta}\right)^{\rho} x^{-\rho(\beta+1)} G^{\rho} (1-G)^{\rho(\lambda-1)} \left[1-(1-G)^{\lambda}\right]^{\rho(a-1)} \left\{1-\left[1-(1-G)^{\lambda}\right]^{a}\right\}^{\rho(b-1)} \quad (34)$$

Applying the binomial expansion of Eq. (12) successively to the three bracketed terms in Eq. (35), with summation indices i, j, and k, yields the series representation

$$[f_{EGTL\text{-}F}(x)]^{\rho} = \left(2ab\beta\lambda\theta^{\beta}\right)^{\rho} x^{-\rho(\beta+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\rho(b-1)}{i}\binom{\rho a-\rho+ai}{j}\binom{\rho\lambda-\rho+\lambda j}{k} \times (-1)^{i+j+k} e^{-2(\rho+k)(\theta/x)^{\beta}} \quad (36)$$

Writing $v_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho(b-1)}{i}\binom{\rho a-\rho+ai}{j}\binom{\rho\lambda-\rho+\lambda j}{k}(-1)^{i+j+k}$, integrating Eq. (37) term by term over $x \in (0,\infty)$ using the same substitution $u = 2(\rho+k)(\theta/x)^{\beta}$ employed in Section 2.4, and collecting the resulting Gamma-function terms, gives

$$\int_0^{\infty}[f_{EGTL\text{-}F}(x)]^{\rho}\,dx = \left(2ab\beta\lambda\theta^{\beta}\right)^{\rho}\frac{\theta^{1-\rho(\beta+1)}}{\beta}\sum_{k=0}^{\infty} v_k\,[2(\rho+k)]^{\frac{1-\rho(\beta+1)}{\beta}}\Gamma\left(\frac{\rho(\beta+1)-1}{\beta}\right) \quad (37)$$

which is finite provided $\rho(\beta+1) > 1$. Substituting Eq. (32) into Eq. (33) gives the Rényi entropy of the EGTL-F distribution as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\left(2ab\beta\lambda\theta^{\beta}\right)^{\rho}\frac{\theta^{1-\rho(\beta+1)}}{\beta}\sum_{k=0}^{\infty} v_k\,[2(\rho+k)]^{\frac{1-\rho(\beta+1)}{\beta}}\Gamma\left(\frac{\rho(\beta+1)-1}{\beta}\right)\right\}, \quad \rho(\beta+1) > 1,\ \rho \neq 1. \quad (38)$$

3.8 Graphical Behaviour of the CDF, PDF, Survival, and Hazard Functions

Fig.s 1–4 display the CDF, PDF, survival function, and hazard function of the EGTL-F distribution for several parameter combinations. The PDF in Fig. 2 shows that the EGTL-F distribution can take unimodal, right-skewed shapes of varying peakedness depending on the parameter values, while the hazard function in Fig. 4 shows that the distribution can accommodate increasing, decreasing, and bathtub-shaped hazard rates, confirming its flexibility for lifetime and reliability applications.

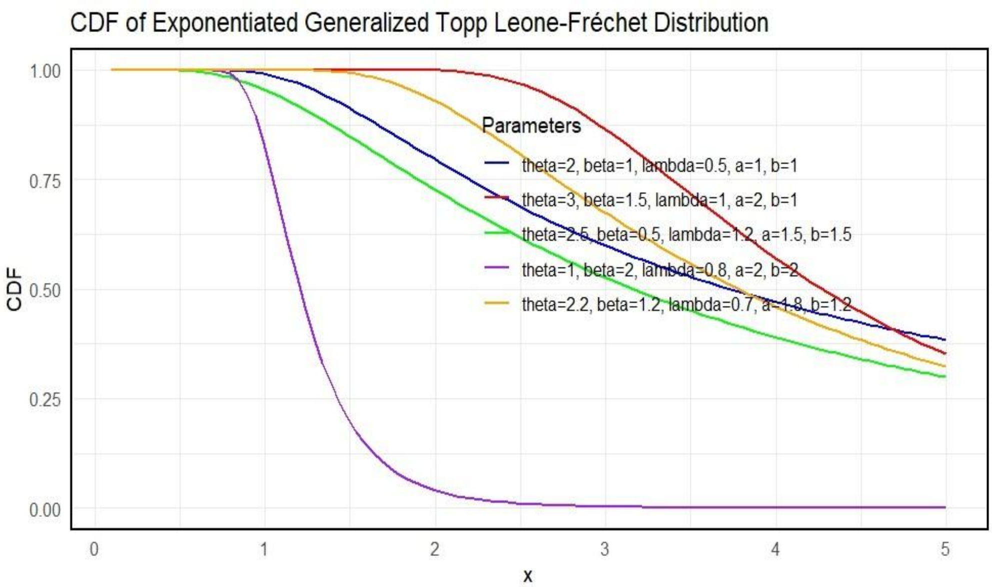

Fig. 1. CDF of the Exponentiated Generalized Topp Leone-Fréchet distribution for several parameter combinations

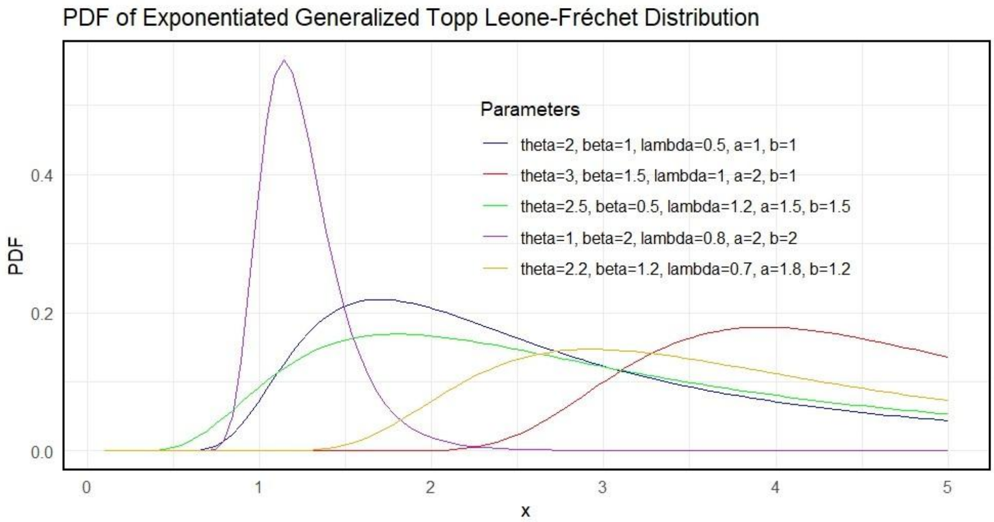

Fig. 2. PDF of the Exponentiated Generalized Topp Leone-Fréchet distribution for several parameter combinations

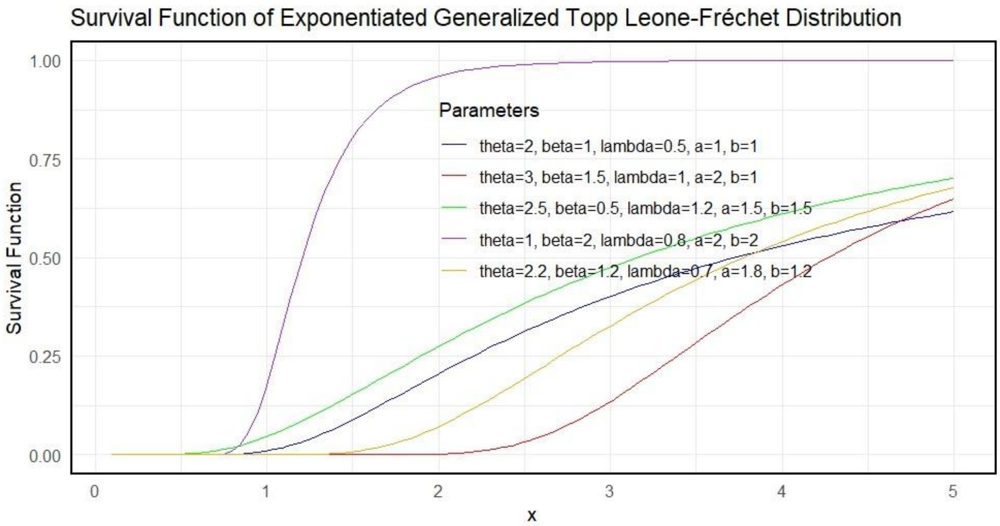

Fig. 3. Survival function of the Exponentiated Generalized Topp Leone-Fréchet distribution for several parameter combinations

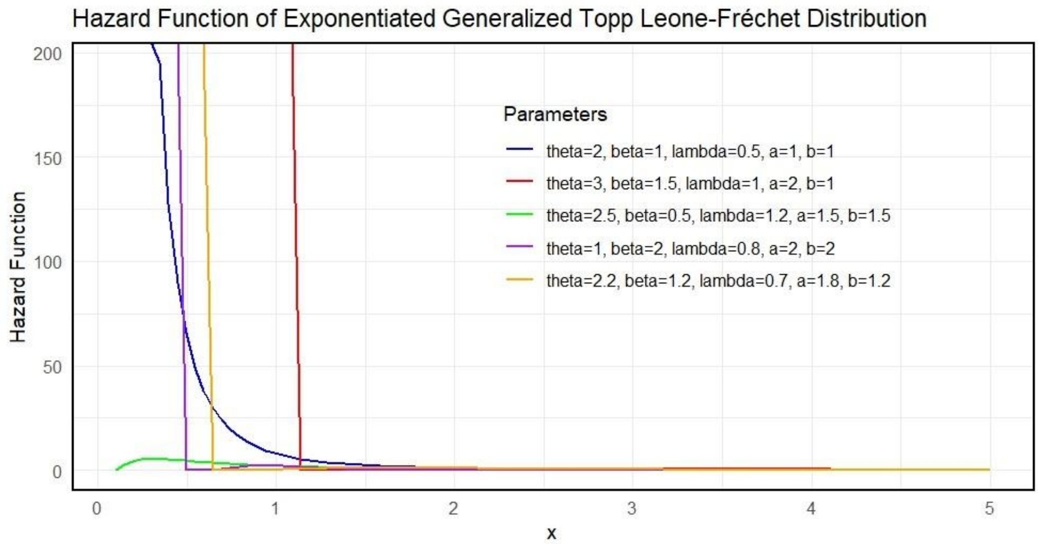

Fig. 4. Hazard function of the Exponentiated Generalized Topp Leone-Fréchet distribution for several parameter combinations

4. RESULT AND DISCUSSION

This section presents simulation results for the maximum likelihood estimators, followed by two real-data applications, with model comparison based on the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Log-Likelihood (LL). The competing distributions used for comparison, the Exponentiated Fréchet distribution (EFD), the Fréchet distribution (FD), and the Exponential distribution (ED), are well-established baseline lifetime models routinely used as benchmarks in distribution-theory papers of this kind; the EFD is the one-parameter exponentiated extension of the Fréchet distribution, FD is the two-parameter Fréchet distribution defined in Eq. (3), and ED is the classical one-parameter Exponential distribution.

4.1 Simulation Study

The performance of the maximum likelihood estimators of the EGTL-F parameters $(a, b, \beta, \lambda, \theta)$ was assessed using a Monte Carlo simulation study. For each of two parameter configurations, random samples of sizes $n = 10$, 50, and 100 were generated from the EGTL-F distribution using the quantile function in Eq. (8) with $r = 1000$ replications. Numerical estimates of the MLEs were obtained using the Nelder–Mead optimization method via the optim function in the stats package of R. For each sample size n and replication $i = 1,2, \dots, r$, the bias and mean squared error (MSE) of each estimator $\hat{\eta}$ were computed as

$$\text{Bias}(\hat{\eta})_{(n)} = \frac{1}{r}\sum_{i=1}^{r}(\hat{\eta}_i - \eta), \qquad \text{MSE}(\hat{\eta})_{(n)} = \frac{1}{r}\sum_{i=1}^{r}(\hat{\eta}_i - \eta)^2 \tag{39}$$

Table 1 reports the results for two separate parameter configurations, labelled Configuration A and Configuration B, used to assess the robustness of the estimators under different true parameter values.

Table 1: Bias and Mean Squared Error (MSE) of MLEs for EGTL-F Parameters
Configuration A: $(a, b, \beta, \lambda, \theta) = (0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5)$

Parameter	Measure	$n = 10$	$n = 50$	$n = 100$
$a = 0.5$	Bias	1.6647	0.2080	0.0461
	MSE	43.4875	1.4625	0.4106
$b = 1.0$	Bias	5.0183	0.6269	0.1505
	MSE	275.3945	10.3821	1.0700
$\beta = 1.5$	Bias	1.9537	0.3744	0.2564
	MSE	20.9866	1.0398	0.4270
$\lambda = 2.0$	Bias	5.1798	1.2944	0.6615
	MSE	385.0507	17.9514	3.1292
$\theta = 2.5$	Bias	4.2560	0.8722	0.3903
	MSE	162.7983	10.4175	1.7559

Configuration B: $(a, b, \beta, \lambda, \theta) = (1.5,\ 2.0,\ 2.5,\ 1.0,\ 1.5)$

Parameter	Measure	$n = 10$	$n = 50$	$n = 100$
$a = 1.5$	Bias	2.1510	0.3246	0.0681
	MSE	102.8908	8.6192	2.1450
$b = 2.0$	Bias	9.4269	2.8908	1.2167
	MSE	884.9778	83.8906	16.9610
$\beta = 2.5$	Bias	3.5880	0.5841	0.3812
	MSE	72.2850	3.2621	1.5495
$\lambda = 1.0$	Bias	6.2034	1.9761	0.7601
	MSE	388.5165	33.3080	6.8574
$\theta = 1.5$	Bias	2.2147	0.9721	0.4316
	MSE	31.2488	6.1774	1.2722

4.2 Application 1: Glass Fibre Strength Data

The first dataset consists of the breaking strengths of 63 strands of 1.5 cm glass fibres, originally measured at the UK National Physical Laboratory (NPL) and widely used in materials-science and reliability studies (Smith & Naylor, 1987; Bourguignon et al., 2014; Merovci et al., 2016). The data are presented in Table 2.

Table 2: Observed Data Set (Strength of 1.5 cm Glass Fibres)

0.55	0.74	0.77	0.81	0.84	1.24	0.93	1.04	1.11	1.13
1.30	1.25	1.27	1.28	1.29	1.48	1.36	1.39	1.42	1.48
1.51	1.49	1.49	1.50	1.50	1.55	1.52	1.53	1.54	1.55
1.61	1.58	1.59	1.60	1.61	1.63	1.61	1.61	1.62	1.62
1.67	1.64	1.66	1.66	1.66	1.70	1.68	1.68	1.69	1.70
1.78	1.73	1.76	1.76	1.77	1.89	1.81	1.82	1.84	1.84
2.00	2.01	2.24							

Table 3 reports the log-likelihood, AIC, and BIC values for the EGTL-F distribution and the three competitor distributions (EFD, FD, ED) fitted to this dataset.

Table 3: Log-Likelihood and Information Criteria (Glass Fibre Strength Data)

S/N	Distribution	Log-Likelihood	AIC	BIC
1	EGTL-F	−63.538	137.076	147.792
2	EFD	−92.745	191.491	197.920
3	FD	−92.745	189.491	193.777
4	ED	−94.930	191.860	194.003

Fig. 5 shows the histogram of the glass fibre strength data together with the fitted density curves of the EGTL-F, EFD, FD, and ED distributions.

4.3 Application 2: Analgesic Relief Time Data

The second dataset, originally reported by Gross and Clark (1975), records the relief times, in minutes, of 20 patients following treatment with an analgesic. The data are presented in Table 4.

Table 4: Relief Times (in Minutes) after Analgesic Treatment

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

Table 5 reports the log-likelihood, AIC, and BIC values for the four fitted distributions on this dataset.

Table 5: Log-Likelihood and Information Criteria (Analgesic Relief Time Data)

S/N	Distribution	Log-Likelihood	AIC	BIC
1	EGTL-F	−28.734	67.474	72.452
2	EFD	−35.166	76.332	79.320
3	FD	−35.166	74.332	76.323
4	ED	−38.000	78.000	78.996

Fig. 6 shows the histogram of the analgesic relief time data together with the fitted density curves of the four distributions.

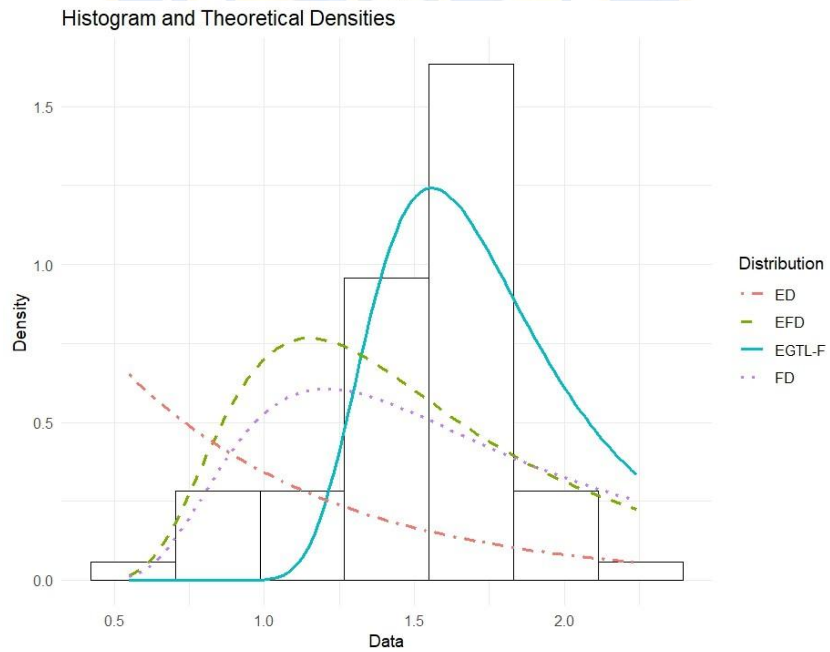

Fig. 5. Histogram of the glass fibre strength data with fitted densities of the EGTL-F, EFD, FD, and ED distributions

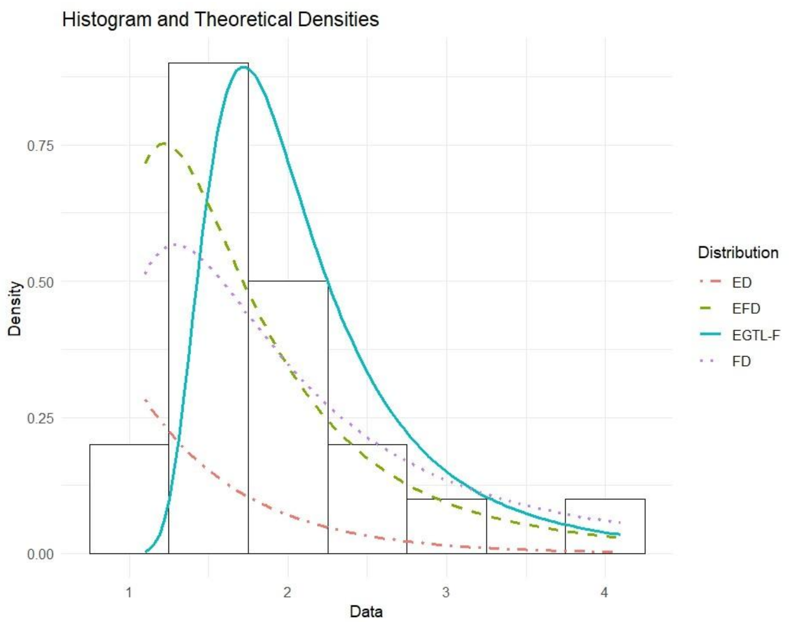

Fig. 6. Histogram of the analgesic relief time data with fitted densities of the EGTL-F, EFD, FD, and ED distributions.

4.4 Interpretation

Table 1 shows that, for both Configuration A and Configuration B, the bias and mean squared error (MSE) of every parameter estimator decrease consistently as the sample size n increases from 10 to 100. This pattern confirms that the maximum likelihood estimators of the EGTL-F distribution are consistent: bias, which measures how far the average estimate departs from the true parameter value, shrinks markedly as more data becomes available, and the MSE, which combines squared bias and variance into a single measure of overall estimation error, falls correspondingly. These results indicate that larger sample sizes yield more accurate and more stable parameter estimates under the EGTL-F model, which is consistent with standard asymptotic theory for maximum likelihood estimation.

The two real-data applications support the practical usefulness of the EGTL-F distribution. For the glass fibre strength data (Table 3), the EGTL-F distribution achieves the highest log-likelihood (-63.538) and the lowest AIC (137.076) and BIC (147.792) among all four fitted distributions, indicating that it achieves a better balance between goodness of fit and model complexity than the EFD, FD, and ED models for this dataset. Similarly, for the analgesic relief time data (Table 5), the EGTL-F distribution again attains the highest log-likelihood (-28.734) and the lowest AIC (67.474) and BIC (72.452), indicating that it captures the pattern of relief times more effectively than the competing models.

These numerical results are supported by the graphical diagnostics in Fig.s 3 and 4. For the glass fibre strength data, the histogram is right-skewed with a high concentration of observations around 1.5, and the fitted EGTL-F density closely follows the empirical histogram across both the central body and the right tail of the distribution, whereas the FD and EFD densities deviate noticeably in the upper tail, and the ED density underestimates the density at small values and fails to capture the right tail at all. A similar pattern is seen for the analgesic relief time data, which is right-skewed with a mode between 1.5 and 2.0 minutes: the EGTL-F density again provides the closest visual fit to both the peak and the tail decay of the histogram, while the EFD and FD densities show visible deviations in the tail, and the ED density captures neither the peak nor the tail well.

Taken together, the simulation results and the two real-data applications provide consistent statistical and graphical evidence that the EGTL-F distribution is an appropriate and competitive model for right-skewed, heavy-tailed lifetime data.

The results of this study are consistent with the theoretical motivation for combining the Topp Leone transformation with the Fréchet distribution. The additional shape parameters a, b, and λ introduced through the EGTL-G family give the EGTL-F distribution greater control over skewness and tail weight than the two-parameter Fréchet distribution or the one-parameter Exponential distribution can offer on their own. This additional flexibility is reflected directly in the consistently better information-criterion values obtained for EGTL-F across both real datasets, and in its visibly closer fit to the empirical histograms in Fig.s 5 and 6.

At the same time, in Section 3, particularly the existence condition for ordinary moments () and the Rényi entropy derivation, are important practically as well as theoretically. A user fitting the EGTL-F distribution to data with an estimated shape parameter close to or below 1 should be cautious about reporting the mean or higher moments, since these would not exist under the fitted model; in such cases, the median (Eq. 9) or other quantile-based summaries provide a more reliable description of central tendency. The Rényi entropy expression in Eq. (36) similarly depends on the condition , which should be checked whenever the entropy is computed for a specific fitted dataset.$r < \beta\hat{\beta}\rho(\beta + 1) > 1$

The simulation results in Table 1, now clearly distinguishing two parameter configurations, show that the bias and MSE patterns observed are not specific to a single, arbitrarily chosen set of true parameter values; both configurations display the same qualitative pattern of decreasing bias and MSE with increasing sample size, which strengthens confidence in the general consistency of the EGTL-F maximum likelihood estimators.

A limitation of the present study is that the two real datasets used, while standard benchmark datasets in the reliability and lifetime-modelling literature, are both moderate in size (63 and 20 observations, respectively). Applying the EGTL-F distribution to larger datasets, and to datasets from other application areas such as finance and environmental science, would further strengthen the evidence for its general usefulness.

5. CONCLUSION

This paper has proposed the Exponentiated Generalized Topp Leone-Fréchet (EGTL-F) distribution, a five-parameter flexible model for asymmetric, heavy-tailed lifetime data, obtained by compounding the EGTL-G family with the Fréchet distribution. We derived its CDF, PDF, survival and hazard functions, quantile function and median, ordinary moments with their existence condition, order statistics, and Rényi entropy, and we estimated its parameters using maximum likelihood. The Monte Carlo simulation study, conducted under two separate parameter configurations, showed that the bias and MSE of the maximum likelihood estimators decrease consistently as the sample size increases from 10 to 100, confirming the consistency of the estimators. Applications to the glass fibre strength data and the analgesic relief time data showed that the EGTL-F distribution achieves a higher log-likelihood and lower AIC and BIC values than the Exponentiated Fréchet, Fréchet, and Exponential distributions, and provides a visibly closer fit to the empirical histograms in both cases. These findings indicate that the EGTL-F distribution is a realistic and competitive alternative to classical lifetime models for describing positively skewed, heavy-tailed data, including data exhibiting abrupt extreme events.

5.1 Recommendation

Based on these findings, the EGTL-F distribution is recommended as a candidate model for positively skewed, heavy-tailed lifetime and reliability data, such as material-strength measurements and treatment-response times. Practitioners intending to use this model are advised to: (i) check the fitted value of the shape parameter β against the moment-existence condition $r < \beta$ before reporting moment-based summaries such as the mean, variance, skewness, or kurtosis; (ii) use the quantile function in Eq. (8) for simulation and for quantile-based inference, since it is now verified to be well defined for all $p \in (0,1)$; and (iii) compare the EGTL-F distribution against a wider range of competitor models and additional real datasets, particularly larger ones, in future applied work to further establish the scope of its usefulness.

5.2 Areas of Further Research

1. **Bayesian estimation.** Future work could develop Bayesian estimation procedures for the EGTL-F parameters, alongside the maximum likelihood approach used here, and compare the two approaches under different prior specifications and sample sizes.
2. **Censored data.** The present study assumes complete data. Extending the estimation procedure to right-censored or interval-censored data would broaden the applicability of the EGTL-F distribution in survival analysis and reliability engineering.
3. **Regression modelling.** Future research could embed the EGTL-F distribution within a regression framework, allowing its scale or shape parameters to depend on covariates, for applications where explanatory variables are available alongside the lifetime outcome.
4. **Bivariate and multivariate extensions.** A bivariate or multivariate EGTL-Fréchet family could be developed for applications where two or more correlated heavy-tailed variables must be modelled jointly, such as joint extreme events in environmental or financial data.
5. **Larger and more diverse datasets.** Future applied work should test the EGTL-F distribution on larger datasets and on data from additional fields, such as finance, hydrology, and insurance, to further establish the generality of the results obtained here.
6. **Further entropy and information measures.** Beyond the Rényi entropy derived in this paper, future work could derive other information-theoretic measures, such as Shannon entropy and Tsallis entropy, for the EGTL-F distribution.

Author Contributions

Conceptualization, J.M.A. and U.M.M.; methodology, J.M.A.; software, J.M.A.; validation, J.M.A., U.M.M. and J.H.A.; formal analysis, J.M.A.; investigation, J.M.A.; resources, J.H.A.; data curation, J.M.A.; writing original draft preparation, J.M.A.; writing review and editing, U.M.M. and J.H.A.; visualization, J.M.A.; supervision, U.M.M.; project administration, J.H.A. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

The two real datasets analysed in this study (the glass fibre strength data and the analgesic relief time data, Tables 2 and 4) are standard published benchmark datasets, originally reported in Smith and Naylor (1987), Bourguignon et al. (2014), Merovci et al. (2016), and Gross and Clark (1975) cited in the text, and are publicly available through those sources.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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