Hybrid Reverse Proxy Architectures: A Copula-Theoretic Perspective on System Reliability

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Abstract

This study explores the reliability of hybrid reverse proxy architectures, which combine distributed and centralized load balancing. Due to complex interdependencies among subsystems, traditional reliability models fall short, especially for correlated or cascading failures. The paper introduces a Gumbel copula-based approach to model the joint failure behavior of these subsystems, effectively capturing the risk of simultaneous extreme failures. The method enables more accurate estimation of complete failure probabilities and supports better fault recovery planning through copula repair strategies, providing a practical framework for improving reliability in complex distributed systems.

Keywords: Availability, Architectures, Copula, Hybrid, Proxy, Reliability, System

1. Introduction

In modern distributed systems, the demand for scalability, performance, and availability has led to the widespread adoption of hybrid reverse proxy architectures. These architectures combine both distributed and centralized load balancing mechanisms to manage traffic flow across application services and backend resources. On one end, distributed reverse proxies often embedded within application instances enable localized routing decisions, increasing responsiveness and autonomy. On the other, centralized load balancers provide system-wide traffic control, policy enforcement, and observability. Together, they create a flexible, layered system capable of handling complex workloads across microservices and databases.

However, this architectural flexibility comes at the cost of increased interdependence among components. The application layer, load balancing layer, and database layer are tightly coupled in ways that can lead to non-linear failure behavior, especially under stress. Traditional reliability models, which often assume component independence or rely on simplistic correlation metrics, are inadequate for capturing the joint failure risks inherent in such interconnected systems. In particular, they struggle to model simultaneous or cascading failures scenarios where multiple subsystems fail together, resulting in total system outages.

To address this gap, this study introduces a copula-theoretic approach for evaluating the reliability of hybrid reverse proxy architectures, with a specific focus on the Gumbel copula. Copula models allow for the separation of marginal failure distributions from their dependence structure, offering a mathematically rigorous way to capture tail dependencies; the likelihood of extreme failures occurring together. The Gumbel copula, in particular, is well-suited for modeling upper-tail dependence, making it ideal for analyzing critical failure events across interdependent layers. The critical aim of this study is to enhance predictive reliability analysis, enable proactive repair strategies, and optimize system design by leveraging the Gumbel copula's ability to model joint failure behavior in complex, layered systems. This allows for more accurate risk assessment, early detection of cascading failure patterns, and improved long-term availability and cost-efficiency in hybrid reverse proxy environments.

This paper is structured as follows: the first section introduces the topic; the second section reviews related works; the third section presents nomenclature, assumptions and the model description; the fourth section details model formulation and solutions; the fifth presents model analysis for specific cases; and the sixth section discusses the results; and the final section concludes the study with key insights.

2. Literature review

Numerous researchers have explored the performance of repairable systems through copula repair techniques, which offer an effective method for modeling dependencies among various components. These methods have demonstrated significant effectiveness in enhancing the reliability and strength of system architectures. For instance, Singh and Kumar (2017) conduct a transient analysis of machining systems that incorporate spare parts provisioning and geometric reneging, providing insights into system behavior during short-term fluctuations and the effects of reneging on system performance. Kumar and Singh (2017) develop an optimal (N, F) policy for queue-dependent, time-sharing redundant machining systems, optimizing system performance by balancing redundancy and workload in dynamic operating conditions. Singh, Kumar, and Sharma (2017) perform a sensitivity analysis on repairable redundant systems experiencing switching failures and geometric reneging, highlighting how key parameters affect system reliability and maintenance strategies.

Kumar, Singh, and Lal (2018) present a reliability model for a fault-tolerant machining system that accounts for reboot and recovery delays after failures. The study analyzes how these delays affect system availability and performance, using probabilistic modeling to provide more realistic reliability predictions for industrial systems. Pundir and Patawa (2019) investigated a repairable system with two dissimilar cold standby units, simulating exponential failure rates alongside arbitrary waiting and repair times. Similarly, Bisht et al. (2020) applied copula theory to analyze repair processes in industrial systems, while Yusuf et al. (2021) focused on reliability analysis of distributed systems by using the Gumbel-Hougaard family copula for joint probability distributions to improve data replication. Poonia (2021) examined a multi-state computer network consisting of database and web servers, exploring copula repair through supplementary techniques and Laplace transforms. Sha et al. (2021) proposed copula-based reliability analyses for hybrid systems, and Singh et al. (2022) introduced a copula linguistic approach to evaluate redundant k-out-of-n: G systems with sequential degradations, as well as assessing multi-computer systems using copula-based repair policies.

In related work, Xian et al. (2022) studied the reliability of k-out-of-n: F systems equipped with multi-state protective devices under shock environments, while Salihu et al. (2022) investigated series-connected computer networks via copula methods. Additionally, Sengar and Ram (2022) applied copula techniques to assess the reliability and performance of complex manufacturing systems featuring inspection facilities. Most recently, Sanusi and Yusuf (2023) analyzed availability and cost-effectiveness of fault-tolerant series-parallel systems integrating human-robotic operators. Building upon this extensive literature, the present study aims to deepen the understanding of how copula modeling can improve the assessment and optimization of network reliability and performance in complex hybrid systems.

2. 1 Contribution to Literature

This study contributes to the literature by introducing a Gumbel copula-theoretic framework for modeling the reliability of hybrid reverse proxy architectures. Unlike traditional models that assume component independence, this approach captures upper-tail dependencies, allowing for realistic analysis of simultaneous and cascading failures. It enhances predictive maintenance, supports targeted repair strategies, and improves early fault detection through sensitivity analysis. Despite initial complexity, the model proves cost-efficient over time, offering a more accurate and resilient method for managing interdependent system failures in distributed computing environments.

The Gumbel copula is chosen for its ability to effectively model asymmetric dependencies, particularly capturing stronger upper-tail dependence which is crucial for assessing joint extreme events in system reliability. Compared to other copulas like the Clayton or Frank, the Gumbel copula better represents scenarios where simultaneous high values (e.g., failures) are more likely. Model fitting and validation involve comparing goodness-of-fit measures and dependence structures to ensure the chosen copula accurately reflects observed data, thereby providing a robust framework for analyzing hybrid reverse proxy architectures.

2.2. Nomenclatures, Assumptions and System Architecture Overview

2. 2. 1 Nomenclatures

1 t: passage time.

S: Variable Laplace notation.

v_1: Rate of failure of application residing in subsystem 1.

v_2: Rate of failure of database server residing in subsystem 3.

v_3: Rate of failure of inbuild load balancer residing in subsystem 1.

v_4: Rate of failure of load balancer residing in subsystem 2.

$\eta(x)/\eta(y)$: Partial failure rate of repair of unit residing in subsystem 1 and 3.

$\eta_0(x)/\eta_0(y)/n_0(m)$: Rate of copula repair for complete or down states for subsystem 1, 2, and 3.

$H_i(y_k,t)$: The chance that the system staying in Si for i =0 to 12.

$\overline{H}_i(s)$: State Laplace transformation.

$E_p(t)$: Profit expectation during the passage of time [0, t).

M_1,M_2: Revenue realized and the corresponding cost of service in passage of time [0, t).

$S_R(y_k): S_R(y_k)=R(y_k)e^{-\int_0^\infty R(y_k)dy_k}$: First shifting property of Laplace transform

$L\left[S_R(y_k)\right]:\ S_R(y_k)=\int_0^\infty e^{-sy_k}R(y_k)e^{-\int_0^\infty R(y_k)}dy_k$: Second shifting property of Laplace transform.

2.2.2 Assumptions

1. Failures among the three primary subsystems (Application servers, External load balancer, Database servers) are not statistically independent but are correlated due to shared infrastructure, cascading effects, or environmental factors.
2. The hybrid system transitions between a finite set of predefined states, each representing a unique combination of degraded subsystems, and transitions occur due to individual or joint component failures or repairs.
3. Subsystem degradation and recovery follow Markovian dynamics, with constant failure and repair rates over time, implying memoryless behavior.
4. Each subsystem has built-in redundancy (e.g., multiple application/database servers), allowing the system to tolerate multiple component failures before becoming non-operational.
5. Failure in Subsystem 2 (external load balancer) results in immediate and total system failure, regardless of the operational status of Subsystems 1 and 3.

2.2.3 System Architecture Overview

The system follows a hybrid architecture combining distributed and centralized reverse proxies, structured into three interconnected subsystems:

1. Subsystem A: Application Layer
   a. Consists of multiple application instances, each with an embedded (distributed) load balancer.
   b. Responsible for handling user requests, business logic, and routing traffic independently.
2. Subsystem B: Load Balancing Layer
   a. Composed of centralized reverse proxies/load balancers between the application and database layers.
   b. Manages global traffic distribution, policy enforcement, and observability.
3. Subsystem C: Database Layer
   a. Houses persistent storage systems (databases), serving as the final destination for application requests.
   b. Can receive traffic from both centralized and distributed routing paths.
4. Failure Modes

The system is vulnerable to two main types of failure:

1. Partial Failures
   a. Affect only specific components (e.g., a single app instance or DB node).
   b. Caused by localized issues like hardware faults or temporary overloads.
   c. Handled via General Repair mechanisms automated recovery processes such as rerouting, restarting, or self-healing.
2. Complete Failures
   a. Involve total breakdown across one or more entire subsystems.
   b. Triggers include cascading failures, widespread outages, or major misconfigurations.

c. Require Copula Repair, a system-wide coordinated recovery process involving infrastructure rebuilds, data restoration, and service synchronization.

5. Critical Conditions Leading to Complete Failure

a. Total Application Layer Failure: All application nodes fail, halting all user transactions.

b. Total Database Layer Failure: All databases become unavailable, making the system unable to process data.

c. Critical Load Balancer Failure: Central load balancers fail, cutting off routing between components despite individual health.

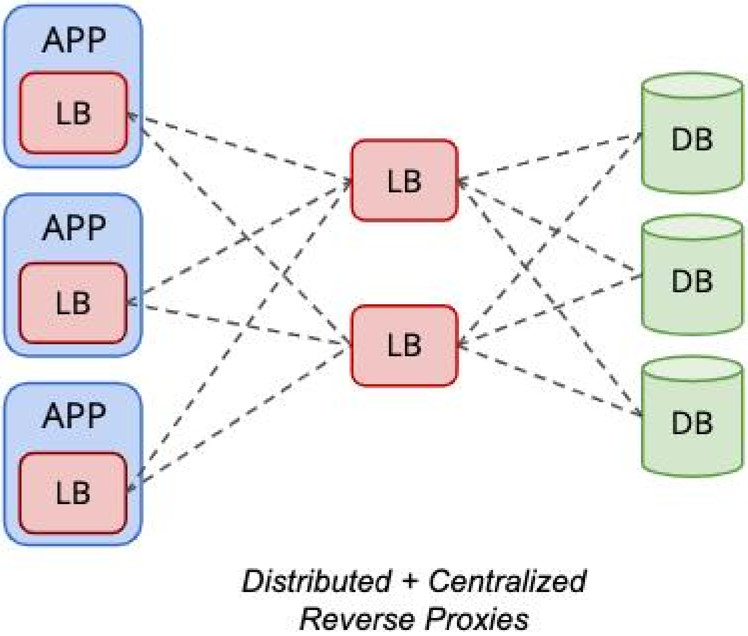

Figure 1: Reverse Proxy Architecture

$\delta_0 = (v_1 + v_3)$

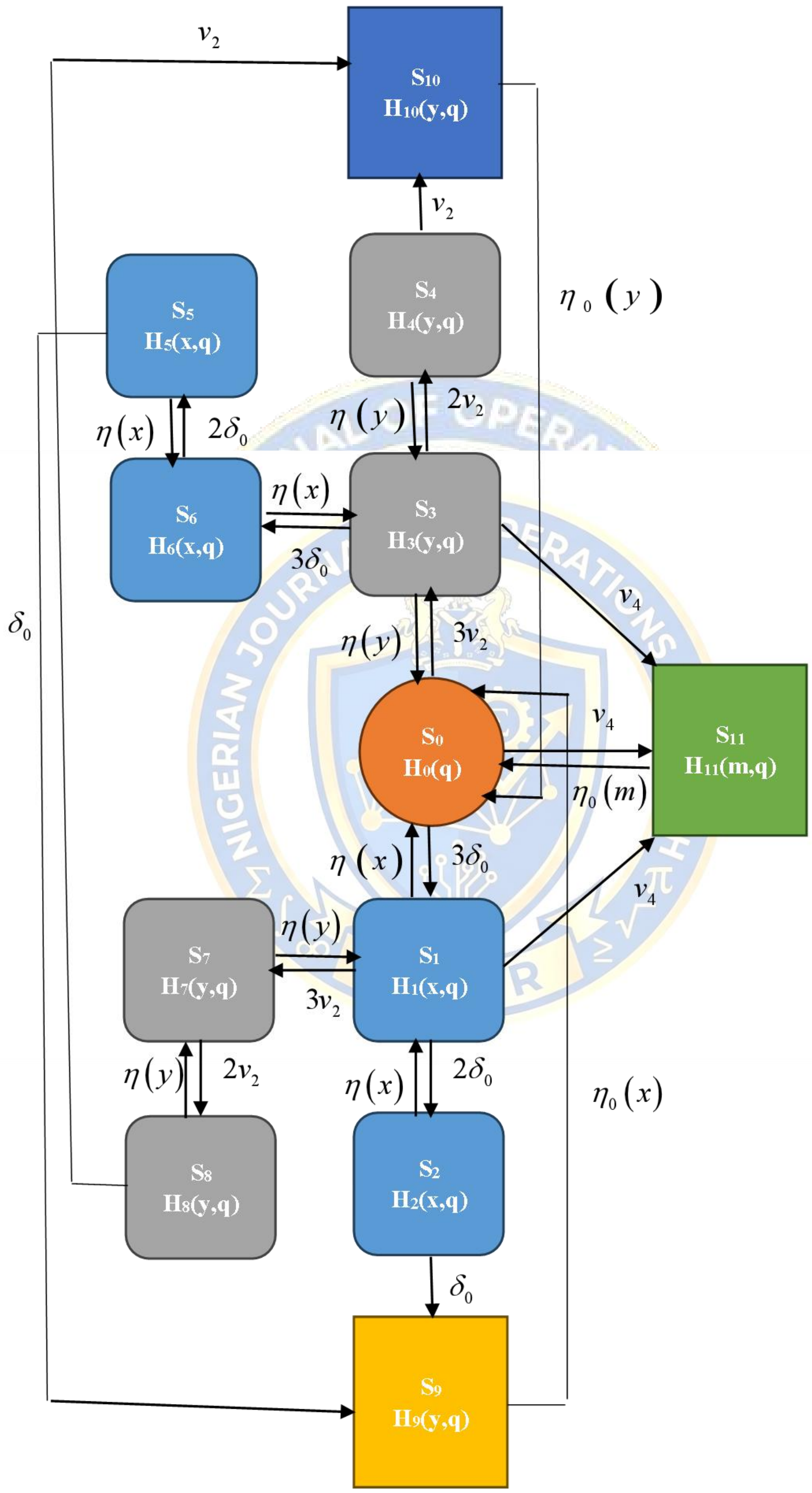

Figure 2: Transition Diagram

3. Methodology

To evaluate the dynamic reliability of a hybrid reverse proxy system, we propose a state-based modeling approach that captures the degradation pathways of individual subsystems and their interactions under failure conditions. This methodology departs from traditional component-wise analysis by considering correlated failures, using copula functions to better characterize joint failure behavior among subsystems.

Our system comprises three primary subsystems:

A. Subsystem 1 – Application servers with integrated load balancing
B. Subsystem 2 – External load balancer
C. Subsystem 3 – Database servers

Each of these can independently degrade due to hardware failures, software faults, or operational errors. However, the interdependence between these subsystems requires a joint reliability model.

State Transition Modeling

We construct a state transition model to represent various operational and failure conditions of the hybrid architecture. The system initially starts in a fully functional state, with all subsystems operating optimally. Each state captures a specific combination of partial failures within the subsystems. A transition from one state to another represents the occurrence of an additional failure in one or more subsystems. The state diagram (Figure 2) illustrates all possible transitions, categorized by failure types and locations.

3.1 Operational States

i. S_0: (Initial State): All subsystems are fully operational; no component has failed. The system is functional.
ii. S_1: A single failure has occurred in the application/load balancer module within Subsystem 1. Subsystems 2 and 3 remain operational.
iii. S_2: A second failure occurs in Subsystem 1. Despite redundancy, the system remains functional.
iv. S_3: A database server in Subsystem 3 fails. Subsystems 1 and 2 are unaffected.
v. S_4: A second database server fails in Subsystem 3. The system continues to operate within the fault-tolerance margin.
vi. S_5: Joint degradation: one failed application/load balancer in Subsystem 1 and one failed database server in Subsystem 3. Subsystem 2 is fully operational.
vii. S_6: Joint degradation: second failure in Subsystem 1 and one failed database server in Subsystem 3. Subsystem 2 still operational.
viii. S_7: One failed application server in Subsystem 1 and two database server failures in Subsystem 3.
ix. S_8: Two failed application servers in Subsystem 1 and two failed database servers in Subsystem 3. Subsystem 2 remains operational. System is on the verge of failure.
x. Failure States
xi. S_9 (Application Path Failure): All application/load balancer units in Subsystem 1 have failed. The system is no longer operational.
xii. S_{10} (Database Path Failure): All database servers in Subsystem 3 are down. System fails due to loss of data backend.

xiii. S_{11} (External Load Balancer Failure): A failure in Subsystem 2 (external load balancer) causes full system failure due to traffic distribution failure.

3.2 Rate-Based Modeling and Differential Equations

From the state transition diagram (Figure 2), we derive a system of differential equations that model the rate of change of probability over time for each state. Each equation considers:

i. The rate at which the system enters a given state from adjacent states (via failure or repair).
ii. The rate at which the system exits that state (due to further degradation or recovery).

Let denote the probability of the system being in state Si at time t The resulting system of equations describes how the system evolves over time under probabilistic failure dynamics.

We define the transition rates (failure rates) and (repair rates), which are parameterized based on subsystem characteristics. These rates can be modified to account for environmental stress, usage intensity, or hardware heterogeneity.

Copula-Theoretic Modeling of Dependency

Traditional reliability models often assume independent failures across subsystems. However, in distributed architectures like reverse proxies, failures are not independent they may be correlated due to shared network infrastructure, cascading effects, or common-mode failures (e.g., power loss, misconfiguration).

To capture these dependencies, we incorporate copula function to model the joint distribution of subsystem failures. Specifically, we use Gumbel Copulas for upper tail dependence (e.g., overload-induced failures). By embedding this copula structure into our probability model, we accurately characterize the likelihood of simultaneous subsystem degradations.

Analytical Solution via Laplace Transforms

To solve the system of differential equations efficiently, we apply the Laplace transform technique, which converts the system from the time domain into the frequency (s-domain). In this domain, differential equations become algebraic equations, simplifying the process of solving for the transformed state probabilities Pi(s).

After solving the algebraic system, we apply the inverse Laplace transform to recover time-domain expressions Pi(t).

3.3 Reliability Metrics Extraction

From the time-domain solutions, we derive key reliability indicators:

i. System Reliability: Probability that the system remains in a non-failure state up to time.
ii. Availability: Probability the system is operational at any arbitrary time t accounting for repair.
iii. Mean Time To Failure (MTTF): The average time a non-repairable system or component operates before it fails.
iv. Sensitivity: It measures the proportion of actual positives that are correctly identified by the model.
v. Cost Function: A mathematical function used to evaluate how well a model's predictions match the actual outcomes.

3.4 Parameter Estimation, Validation, and Dependency Assumptions

a. Parameter Estimation: In the copula-theoretic framework, parameter estimation involves calibrating both marginal distributions (e.g., failure rates of individual reverse proxies or components) and the copula parameters that define dependency structures among subsystems. This is typically done using observed uptime/downtime data or simulated failure traces.

b. Simulation Validation: Simulation models of the hybrid reverse proxy system are validated by comparing synthetic reliability outcomes—generated under the estimated copula model—with empirical reliability data. This ensures the model captures real-world dependency patterns and failure propagation accurately.

c. Assumptions About Subsystem Dependencies: The copula-based approach explicitly models dependencies among subsystems (e.g., load balancers, caching layers, and reverse proxies). Assumptions may include tail dependencies, asymmetric failure correlations, or conditional independence, depending on the copula family used (e.g., Clayton for lower tail dependence). These assumptions critically affect the joint reliability analysis of the architecture.

3.5 Model Formulation

The differential equations presented below are derived from the transition diagram (Figure 2) above through a probabilistic method.

$$\left\{\frac{\partial}{\partial q}+\delta_0+3v_2+v_3\right\}H_0(q)=\int_0^\infty \eta(x)H_1(x,q)\,dx+\int_0^\infty \eta(y)H_3(y,q)\,dy+\int_0^\infty \eta_0(x)H_9(x,q)\,dx+\int_0^\infty \eta_0(y)H_{10}(y,q)\,dy+\int_0^\infty \eta_0(m)H_{11}(m,q)\,dm \tag{1}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+2\delta_0+3v_2+v_4+\eta(x)\right\}H_1(x,q)=0 \tag{2}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+\delta_0+\eta(x)\right\}H_2(x,q)=0 \tag{3}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+3\delta_0+2v_2+v_4+\eta(y)\right\}H_3(y,q)=0 \tag{4}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+v_2+\eta(y)\right\}H_4(y,q)=0 \tag{5}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+\delta_0+\eta(x)\right\}H_5(x,q)=0 \tag{6}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+2\delta_0+\eta(x)\right\}H_6(x,q)=0 \tag{7}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+2v_2+\eta(y)\right\}H_7(y,q)=0 \tag{8}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+v_2+\eta(y)\right\}H_8(y,q)=0 \tag{9}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+\eta_0(x)\right\}H_9(x,q)=0 \tag{10}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+\eta_0(y)\right\}H_{10}(y,q)=0 \tag{11}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial m}+\eta_0(m)\right\}H_{11}(m,q)=0 \tag{12}$$

3.5.1 Boundary conditions

The boundary conditions given below are also derived from the transition diagram shown in Figure 2 above.

$$H_1(0,q)=3\delta_0 H_0(q) \tag{13}$$

$$H_2(0,q)=6\delta_0^2 H_0(q) \tag{14}$$

$$H_3(0,q)=3v_2 H_0(q) \tag{15}$$

$$H_4(0,q)=6v_2^2 H_0(q) \tag{16}$$

$$H_5(0,q)=18\delta_0^2 v_2 H_0(q) \tag{17}$$

$$H_6(0,q)=9\delta_0 v_2 H_0(q) \tag{18}$$

$$H_7(0,q)=9\delta_0 v_2 H_0(q) \tag{19}$$

$$H_8(0,q)=18\delta_0 v_2^2 H_0(q) \tag{20}$$

$$H_9(0,q)=6\delta_0^3(1+3v_2)H_0(q) \tag{21}$$

$$H_{10}(0,q)=6v_2^3(1+3\delta_0)H_0(q) \tag{22}$$

$$H_{11}(0,q)=v_4(3v_2v_5+v_3+3\delta_0)H_0(q) \tag{23}$$

3.5.2 Model Solutions

The Laplace transform of the equations (1) to (12) are obtained and given as:

$$\{s+\delta_0+3v_2+v_3\}\overline{H_0}(s)=\int_0^\infty \eta(x)\overline{H_1}(x,s)dx+\int_0^\infty \eta(y)\overline{H_3}(y,s)dy+\int_0^\infty \eta_0(x)\overline{H_9}(x,s)dx+\int_0^\infty \eta_0(y)\overline{H_{10}}(y,s)dy+\int_0^\infty \eta_0(m)\overline{H_{11}}(m,s)dm \tag{24}$$

$$\left\{s+\frac{\partial}{\partial x}+2\delta_0+3v_2+v_4+\eta(x)\right\}\overline{H_1}(x,s)=0 \tag{25}$$

$$\left\{s+\frac{\partial}{\partial x}+\delta_0+\eta(x)\right\}\overline{H_2}(x,s)=0 \tag{26}$$

$$\left\{s+\frac{\partial}{\partial y}+3\delta_0+2v_2+v_4+\eta(y)\right\}\overline{H_3}(y,s)=0 \tag{27}$$

$$\left\{s+\frac{\partial}{\partial y}+v_2+\eta(y)\right\}\overline{H_4}(y,s)=0 \tag{28}$$

$$\left\{s+\frac{\partial}{\partial x}+\delta_0+\eta(x)\right\}\overline{H_5}(x,s)=0 \tag{29}$$

$$\left\{s+\frac{\partial}{\partial x}+2\delta_0+\eta(x)\right\}\overline{H_6}(x,s)=0 \tag{30}$$

$$\left\{s+\frac{\partial}{\partial y}+2v_2+\eta(y)\right\}\overline{H_7}(y,s)=0 \tag{31}$$

$$\left\{s+\frac{\partial}{\partial y}+v_2+\eta(y)\right\}\overline{H_8}(y,s)=0 \tag{32}$$

$$\left\{s+\frac{\partial}{\partial x}+\eta_0(x)\right\}\overline{H_9}(x,s)=0 \tag{33}$$

$$\left\{s+\frac{\partial}{\partial y}+\eta_0(y)\right\}\overline{H_{10}}(y,s)=0 \tag{34}$$

$$\left\{s+\frac{\partial}{\partial m}+\eta_0(m)\right\}\overline{H_{11}}(m,s)=0 \tag{35}$$

The Laplace transforms of equations (13) to (23) are given as:

$$\overline{H_1}(0,s)=3\delta_0\overline{H_0}(s) \tag{36}$$

$$\overline{H_2}(0,s)=6\delta_0^2\overline{H_0}(s) \tag{37}$$

$$\overline{H_3}(0,s)=3v_2\overline{H_0}(s) \tag{38}$$

$$\overline{H_4}(0,s)=6v_2^2\overline{H_0}(s) \tag{39}$$

$$\overline{H_5}(0,s)=18\delta_0^2v_2\overline{H_0}(s) \tag{40}$$

$$\overline{H_6}(0,s)=9\delta_0v_2\overline{H_0}(s) \tag{41}$$

$$\overline{H_7}(0,s)=9\delta_0v_2\overline{H_0}(s) \tag{42}$$

$$\overline{H_8}(0,s)=18\delta_0v_2^2\overline{H_0}(s) \tag{43}$$

$$\overline{H_9}(0,s)=6\delta_0^3(1+3v_2)\overline{H_0}(s) \tag{44}$$

$$\overline{H_{10}}(0,s)=6v_2^3(1+3\delta_0)\overline{H_0}(s) \tag{45}$$

$$\overline{H_{11}}(0,s)=v_4(3v_2+v_3+3\delta_0)\overline{H_0}(s) \tag{46}$$

Solving equations (24) to (35) with the help of their boundary conditions presented in equations (36) to (46), we obtain the following subsequent equations:

$$\overline{H_0}(s)=\frac{1}{R(s)}, \tag{47}$$

$$\overline{H_1}(s)=\frac{3\delta_0}{R(s)}\left(\frac{1-\overline{S_\eta}(s+2\delta_0+3v_2+v_5)}{(s+2\delta_0+3v_2+v_5)}\right), \tag{48}$$

$$\overline{H_2}(s)=\frac{6\delta_0^2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+\delta_0)}{(s+\delta_0)}\right), \tag{49}$$

$$\overline{H_3}(s)=\frac{3v_2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+3\delta_0+2v_2+v_5)}{(s+3\delta_0+2v_2+v_5)}\right), \tag{50}$$

$$\overline{H_4}(s)=\frac{6v_2^2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+v_2)}{(s+v_2)}\right), \tag{51}$$

$$\overline{H_5}(s)=\frac{18\delta_0^2 v_2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+\delta_0)}{(s+\delta_0)}\right), \tag{52}$$

$$\overline{H_6}(s)=\frac{9\delta_0 v_2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+2\delta_0)}{(s+2\delta_0)}\right), \tag{53}$$

$$\overline{H_7}(s)=\frac{9\delta_0 v_2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+2v_2)}{(s+2v_2)}\right), \tag{54}$$

$$\overline{H_8}(s)=\frac{18\delta_0 v_2^2}{R(s)}\left(\frac{1-\overline{S_\eta}(s+v_2)}{(s+v_2)}\right), \tag{55}$$

$$\overline{H_9}(s)=\frac{6\delta_0^2(1+3v_2)}{R(s)}\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right), \tag{56}$$

$$\overline{H_{10}}(s)=\frac{6v_2^2(1+3\delta_0)}{R(s)}\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right), \tag{57}$$

$$\overline{H_{11}}(s)=\frac{v_4(3v_2+v_3+3\delta_0)}{R(s)}\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right), \tag{58}$$

To determine $H_{up}(t)$, we add up the probabilities of all active states.

$$H_{up}(s)=H_0(s)+H_1(s)+H_2(s)+H_3(s)+H_4(s)+H_5(s)+H_6(s)+H_7(s)+H_8(s) \tag{59}$$

$$\overline{H_{up}}(s)=\frac{1}{R(s)}\left\{\begin{array}{l}1+3\delta_0\left(\frac{1-\overline{S_\eta}(s+2\delta_0+3v_2+v_5)}{(s+2\delta_0+3v_2+v_4)}\right)+6\delta_0^2\left(\frac{1-\overline{S_\eta}(s+\delta_0)}{(s+\delta_0)}\right)+\\ 3v_2\left(\frac{1-\overline{S_\eta}(s+3\delta_0+2v_2+v_4)}{(s+3\delta_0+2v_2+v_4)}\right)+6v_2^2\left(\frac{1-\overline{S_\eta}(s+v_2)}{(s+v_2)}\right)+\\ 18\delta_0^2v_2\left(\frac{1-\overline{S_\eta}(s+\delta_0)}{(s+\delta_0)}\right)+9\delta_0v_2\left(\frac{1-\overline{S_\eta}(s+2\delta_0)}{(s+2\delta_0)}\right)+\\ 9\delta_0v_2\left(\frac{1-\overline{S_\eta}(s+2v_2)}{(s+2v_2)}\right)+18\delta_0v_2^2\left(\frac{1-\overline{S_\eta}(s+v_2)}{(s+v_2)}\right)\end{array}\right\}. \tag{60}$$

Where,

$$R(s)=(s+3\delta_0+3v_2+v_3)-\left\{\begin{array}{l}3\delta_0\left(1-\overline{S_\eta}(s+2\delta_0+3v_2+v_4)\right)+3v_2\left(1-\overline{S_\eta}(s+3\delta_0+2v_2+v_4)\right)+\\ 6\delta_0^2(1+3v_2)\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right)+6v_2^2(1+3\delta_0)\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right)+\\ v_4(3v_2+v_3+3\delta_0)\left(\frac{1-\overline{S_{\eta_0}}(s)}{(s)}\right)\end{array}\right\}$$

6. Model Analysis for specific cases

6. 1 System Availability Analysis

Here, the availability via Copula and general repair policy is analysed.

6. 1. 1 Availability via Copula Repair Policy

In this case, we let $S_{\eta_0}(s)=\bar{S}_{exp[x^\theta+\{\log\varphi(x)\}^\theta]^{1/\theta}}(s)=\frac{exp[x^\theta+\{\log\varphi(x)\}^\theta]^{1/\theta}}{s+exp[x^\theta+\{\log\varphi(x)\}^\theta]^{1/\theta}}$, and $\bar{S}_\eta(s)=\frac{\eta}{s+\eta}$. The failure rates are specified as $v_1=0.01$, $v_2=0.02$, $v_3=0.03$ and $v_4=0.04$ with $\eta=x=y=m=1$ and all repair rates set to 1, i.e., $\eta(x)=\eta(y)=\eta_0(x)=\eta_0(y)=\eta_0(m)=1$ in equation (60). Using the inverse Laplace transform, we derive the availability equation for Copula repair as follows:

$$H_{up}(t)=\left\{\begin{array}{l}0.0194e^{-2.773850301t}-0.2724e^{-1.080000000t}-0.0012e^{-1.020000000t}-0.0081e^{-1.040000000t}\\ -0.0332e^{-1.335604406t}-0.0001e^{-1.192789417t}+1.0261e^{-0.1605587649t}\end{array}\right\} \tag{61}$$

6. 1.2 Availability via General Repair Policy

By letting $\bar{S}_\eta(s)=\frac{\eta}{s+\eta}$ in equation (60) and differentiating between parameters with values such as $v_1=0.01$, $v_2=0.02$, $v_3=0.03$ and $v_4=0.04$ with $\eta=1$ and then applying the inverse Laplace transform, we deduce the availability equation for General repair as:

$$H_{up}(t)=\begin{Bmatrix} 0.00004e^{-1.192971422t}-0.0216e^{-1.040000000t}-0.0134e^{-1.365554861t}-0.0037e^{-1.080000000t} \\ +0.0372e^{-1.025855339t}-0.9975e^{-0.01561837901t}+0.0039e^{-1.020000000t} \end{Bmatrix} \quad (62)$$

Using various values for time $t = 0,1,2,3,4,5,6,7,8,9,10.$ Table 1 showcase the system's availability when the repair follows both Copula and General repair policies.

Table 1: Impact of Time on System Availability: A Comparison between Copula and General Repair Policies

Time(s)	0	1	2	3	4	5	6	7	8	9	10
Avail. Via Copula Repair	1.0000	0.9979	0.9899	0.9767	0.9619	0.9468	0.9318	0.9170	0.9024	0.8880	0.8739
Avail. Via General Repair	1.0000	0.9845	0.9682	0.9525	0.9374	0.9227	0.9083	0.8942	0.8804	0.8667	0.8533

6.2 Reliability Analysis

In this case, no repair is given to the system's malfunctioning unit, and hence, all the repair variables in equation (60) are set to zero, and with the inverse Laplace transform, the reliability expression is given below

$$R(t)=\begin{Bmatrix} 1.0396e^{-0.06019754143t}-0.0085e^{-1.040000000t}-0.0028e^{-1.080000000t} \\ -0.0270e^{-1.346966110t}-0.00009e^{-1.192836348t}-0.0013e^{-1.020000000t} \end{Bmatrix} \quad (63)$$

Varying time t as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 on system reliability, the impact is presented in Table 2.

Table 2: Impact of time on system reliability

Time(s)	0	1	2	3	4	5	6	7	8	9	10
Reliability	1.0000	0.9674	0.9183	0.8668	0.8168	0.7692	0.7244	0.6821	0.6423	0.6047	0.5694

6.3 Mean Time To Failure (MTTF) Analysis

If the repair facility is not available, all the repairs in equation (60) will be zero. Taking limit as s tends to zero, we obtain the MTTF as given below:

$$MTTF=\lim_{s\to 0}\frac{1}{(3\delta_0+3v_2+v_3)}\left\{1+\frac{3\delta_0}{2\delta_0+3v_2+v_4}+6\delta_0+\frac{3v_2}{3\delta_0+2v_2+v_4}+6v_2+36\delta_0 v_2+\frac{9}{2}\delta_0+\frac{9}{2}v_2\right\}. \quad (64)$$

When $v_1=0.01$, $v_2=0.02$, $v_3=0.03$ $v_4=0.04$ are fixed, and v_1, v_2, v_3, and v_4 are varied one at a time as 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09 in equation (64), Table 3 presents the variation of MTTF with regard to failure rates.

Table 3: Computation of MTTF corresponding to the different failure rate values

Failure rates	MTTF (ν_2)	MTTF (ν_4)	MTTF (δ_0)
0.01	13.1519	17.0763	18.7847
0.02	11.8685	18.7143	16.2620
0.03	10.9485	14.2545	14.4857
0.04	10.2457	13.1519	13.1519
0.05	9.6851	12.2005	12.1100
0.06	9.2237	11.3718	11.2727
0.07	8.8352	12.4719	10.5847
0.08	8.5023	10.0006	10.0093
0.09	8.2130	9.4276	9.5209
0.10	7.9588	8.9145	9.1012

6.4 Sensitivity Analysis

The sensitivity of this study is obtained by taking partial derivative of the Mean Time To Failure (MTTF) with respect to the system failure rates. The MTFF sensitivity results are shown in Table 4 below by using the set of parameters as $\nu_1 = 0.01$, $\nu_2 = 0.02$, $\nu_3 = 0.03$ $\nu_4 = 0.04$ in the partial differentiation of MTTF.

Table 4: Computation of MTTF sensitivity against the values of failure rate

Failure rates	Sensitivity (ν_2)	Sensitivity (ν_4)	Sensitivity (δ_0)
0.01	-154.8472	-297.9714	-308.5919
0.02	-106.8026	-237.1339	-207.6962
0.03	-79.4854	-190.7776	-152.1936
0.04	-62.2636	-154.8472	-116.9486
0.05	-50.5576	-126.5825	-92.8387
0.06	-42.1431	-104.0581	-75.5334
0.07	-35.8328	-85.9024	-62.6668
0.08	-30.9432	-71.1194	-52.8326
0.09	-27.0552	-58.9736	-45.1446
0.10	-23.8989	-48.9139	-39.0200

6.5 Cost Analysis

If the service facility is always accessible, then the equation below will obtain the expected profit during the interval [0, t).

$$E_p(t) = M_1 \int_0^{\infty} H_{up}(t)dt - M_2(t). \quad (65)$$

where M_1 and M_2 in the interval [0, t) represents the revenue generated and service cost per unit time.

6.5. 1 Cost Analysis via Copula Repair Policy

Equation (66) is obtained for the same set of parameters in equation (61).

$$E_p(t)=M_1\begin{Bmatrix}0.0025e^{-1.080000000t}+0.0078e^{-1.040000000t}+0.0012e^{-1.020000000t}-0.0012e^{-2.773850301t}\\+0.0248e^{-1.335604406t}+0.0001e^{-1.192789417t}-63.9050e^{-0.01605587649t}+63.8755\end{Bmatrix}-M_2t \quad (66)$$

Letting $M_1=1$, $M_2=0.6,0.5,0.4,0.3,0.2,0.1,$ and varying t units of time as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 respectively, the expected benefit results are obtained and shown in Table 5.

Table 5: Expected Profit against Time for Copula Repair Policy

Time	Expected Profit $M_2=0.6$	Expected Profit $M_2=0.5$	Expected Profit $M_2=0.4$	Expected Profit $M_2=0.3$	Expected Profit $M_2=0.2$	Expected Profit $M_2=0.1$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.3985	0.4985	0.5985	0.6985	0.7985	0.8985
2	0.7931	0.9931	1.1931	1.3931	1.5931	1.7931
3	1.1766	1.4766	1.7766	2.0766	2.3766	2.6766
4	1.5460	1.9460	2.3460	2.7460	3.1459	3.5460
5	1.9003	2.4003	2.9003	3.4003	3.9003	4.4003
6	2.2396	2.8396	3.4396	4.0396	4.6396	5.2396
7	2.5639	3.2639	3.9639	4.6639	5.3639	6.0639
8	2.8736	3.6736	4.4736	5.2736	6.0736	6.8736
9	3.1687	4.0687	4.9687	5.8687	6.7687	7.6687
10	3.4496	4.4496	5.4496	6.4496	7.4496	8.4496

6.5. 2 Cost Analysis via General Repair Policy

Using same inputs (parameters) as equation (62), the expected profit for General repair policy is shown below.

$$E_p(t)=M_1\begin{Bmatrix}0.0034e^{-1.080000000t}+0.0207e^{-1.040000000t}-0.0039e^{-1.020000000t}+0.0098e^{-1.365554861t}\\-0.00004e^{-1.192971422t}-0.0363e^{-1.025855339t}-63.8693e^{-0.01561837901t}+63.8754\end{Bmatrix}-M_2t \quad (67)$$

Table 6 below presents various benefit results using different values of the time variable, such as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and the inverse Laplace transform on equation (67).

Table 6: Expected Profit against Time for General Repair Policy

Time	Expected Profit $M_2=0.6$	Expected Profit $M_2=0.5$	Expected Profit $M_2=0.4$	Expected Profit $M_2=0.3$	Expected Profit $M_2=0.2$	Expected Profit $M_2=0.1$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.3925	0.4925	0.5925	0.6925	0.7925	0.8925
2	0.7688	0.9688	1.1688	1.3688	1.5688	1.7688
3	1.1291	1.4291	1.7291	2.0291	2.3291	2.6291
4	1.4739	1.8739	2.2739	2.6739	3.0739	3.4739
5	1.8039	2.3039	2.8039	3.3039	3.8039	4.3039
6	2.1194	2.7194	3.3194	3.9194	4.5194	5.1194
7	2.4207	3.1207	3.8207	4.5207	5.2207	5.9207
8	2.7080	3.5080	4.3089	5.1080	5.9080	6.7080
9	2.9815	4.8815	4.7815	5.6815	6.5815	7.4815
10	3.2415	4.2415	5.2415	6.2415	7.2415	8.2415

7. **In-Depth Analysis: A Copula-Theoretic Perspective on Hybrid Reverse Proxy Architectures**

In this section, we present an in-depth analysis of the results presented above, framed within the context of Hybrid Reverse Proxy Architectures and interpreted through a Copula-theoretic lens, specifically leveraging the Gumbel Copula. The objective is to derive meaningful conclusions that directly inform the design, reliability, and operational strategies of such architectures. By examining the implications of each data set, we aim to evaluate how different repair strategies and system variables impact performance metrics central to high-reliability computing environments.

Table 1 illustrates the relationship between time and system availability under two repair strategies; Copula-based and General applied to a hybrid reverse proxy architecture. The results reveal a consistent decline in availability over time for both policies; however, the Gumbel Copula-based repair policy maintains a significantly higher level of system uptime. This improvement stems from the Gumbel Copula's ability to model upper tail dependence, which captures the likelihood of extreme joint failures in tightly coupled system components. In the operational context of reverse proxy systems where load balancing, traffic routing, and backend service continuity are crucial this allows for better anticipation of simultaneous or cascading component failures, enabling more strategic and targeted repairs. For system architects managing hybrid infrastructures, such modeling translates into fewer service interruptions, improved request handling, and reduced service-level degradations.

Table 2 presents the correlation between time and system reliability, emphasizing natural degradation in hybrid reverse proxy systems when preventive maintenance is absent. Due to the architectural complexity spanning multiple backend services, caching layers, and routing rules minor anomalies can escalate into major failures. By applying the Gumbel Copula, which is particularly effective in capturing asymmetric dependency structures (especially where high-risk failures are more correlated than minor faults), the model allows for predictive maintenance that targets high-risk interdependent components. This enables engineers to proactively schedule interventions before reliability drops sharply, thus preserving system stability in dynamic traffic environments.

Table 3 provides MTTF values across various failure rates, shedding light on how increased component fragility impacts overall system resilience. Under a Gumbel Copula-theoretic framework, the joint distribution of failure times is modelled with an emphasis on capturing the increased probability of multiple high-severity failures occurring simultaneously. This is especially pertinent in hybrid reverse proxy systems, where failures in critical nodes such as DNS resolvers, load balancers, or SSL terminators are not independent. The Gumbel Copula's focus on tail dependence enhances risk quantification in these worst-case scenarios, leading to more accurate reliability assessments and guiding reinforcement strategies for high-impact components.

To assess component criticality, Table 4 performs a sensitivity analysis using variable failure rates. Results indicate decreasing sensitivity as failure rates grow highlighting a non-linear degradation pattern. The Gumbel Copula, due to its ability to model non-linear and asymmetric interdependencies, proves instrumental in detecting early-stage signs of systemic stress. It captures how certain components (e.g., caching layers or health-check modules) disproportionately contribute to global system degradation once a failure threshold is crossed. This early-warning capability is vital in large-scale deployments where small inefficiencies can snowball into critical performance losses.

Tables 5 and 6 investigate the cost-time relationship under Copula-based and General repair policies. While both show that prolonged repair times lead to increased costs, Gumbel Copula-based strategies are more expensive due to their analytical complexity and dependency modeling overhead. However, they offer superior long-term cost efficiency: reduced unplanned downtime, improved failure anticipation, and enhanced repair targeting result in lower cumulative operational expenditure. Additionally, an observed decrease in total costs during peak service periods suggests that optimized resource scheduling enabled by the predictive capabilities of the Copula model improves overall system economics. For mission-critical hybrid reverse proxy deployments, such as those supporting real-time services or transactional systems, the investment in Gumbel Copula-driven repair frameworks is justified by the long-term gains in reliability and performance stability.

8. Conclusion

The application of a Gumbel Copula-theoretic framework to Hybrid Reverse Proxy Architectures offers significant advantages in modeling, predicting, and reducing failures within complex, interdependent systems. By analyzing availability trends, reliability degradation, mean time to failure (MTTF), sensitivity dynamics, and cost-performance balance, several critical insights emerge that inform both operational strategy and system design.

Gumbel Copula-based repair strategies consistently outperform general approaches by maintaining higher system availability. This is primarily due to the model's ability to capture upper tail dependencies, which reflect the increased likelihood of extreme, simultaneous failures across tightly coupled components. In the context of hybrid reverse proxies where disruptions can rapidly propagate from backend latency to frontend request handling this enables more proactive and targeted repair actions, reducing service disruption.

Furthermore, the Gumbel Copula's capacity to model asymmetric and non-linear dependencies enhances predictive maintenance capabilities. It allows for the early identification of cascading failure patterns before they result in significant performance degradation, enabling timely interventions that help preserve long-term system reliability.

From a reliability engineering perspective, the Gumbel Copula provides more realistic joint failure modeling compared to traditional independence-based approaches. This facilitates more accurate risk assessment and the strategic reinforcement of high impact nodes such as load balancers, DNS resolvers, and SSL terminators components whose failures often have system wide consequences.

The sensitivity analysis reveals that system responsiveness to repair diminishes beyond a certain failure rate threshold. This finding reinforces the importance of early fault detection and mitigation. Effective monitoring and intervention prior to reaching these thresholds contribute significantly to the system's long-term stability and adaptability.

Lastly, while Copula-based modeling especially with the Gumbel Copula introduces higher initial complexity and cost, it proves cost-efficient over time. By minimizing unplanned downtime, reducing the frequency of severe service-level degradations, and enabling more efficient resource allocation, these models contribute to lower overall lifecycle costs and improved service continuity.

9. Limitations and Future Research Scope

The current copula-theoretic framework assumes fixed dependency structures and relies on high quality data, which may limit its accuracy in dynamic, real-world settings. The repair model is also simplified, lacking considerations for resource constraints and complex repair interactions. Future work should explore more flexible dependency models, adaptive parameter estimation, and realistic repair strategies, alongside validation using real-world system data.

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