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A Taylor-series-based Integrating-factor approach for a Fuzzy Inventory Model with Two-parameter Weibull Deterioration and Time-dependent Quartic Demand

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ABSTRACT

In this paper, we develop a Taylor-Series-Based Integrating-Factor Approach for a Fuzzy Inventory Model with Two-Parameter Weibull Deterioration and Time-Dependent Quartic. In the model, demand was assumed to be time dependent quartic function. Shortages were not allowed and the demand and Weibull parameters were considered as fuzzy. The Taylor series expansion is used to solve the integrals that arise during the integrating-factor solution, with an adaptive convergence rule applied to control the accuracy of the series approximation. The model determines the optimal order quantity, cycle time, total inventory cost, and average cost. A numerical studies and sensitivity analysis are carried out to examine the effect of changes in parameters on the performance of the inventory system. Finally, the proposed model is compared to the fourth-order Runge-Kutta technique, and the result shows excellent numerical agreement between the two-solution approaches.

1. Introduction

Deterioration is one of the important parts of many inventory items were product experiences declining quality, usefulness, or economic worth with respect to time during storage. Deterioration may arise due to spoilage, evaporation, obsolescence, or any other loss occurring while storing the item. Therefore, deterioration becomes necessary while determining the inventory policies for the items which have varying levels of availability and worth with respect to time. The traditional EOQ (Economic Order Quantity) model developed by Harris (1913) formed the basis for optimal inventory management when the demand was deterministic. However, the assumption of constant demand and non-deteriorating inventories is highly restrictive for many real-world inventory management situations. Deterioration was considered in inventory management by Ghare and Schrader (1963), while Covert and Philip (1973) and Philip (1974) further made developments in the field of deterioration models having time-varying characteristics. Demand is also another important factor in inventory systems

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In practice, the demand rate fluctuates over time, rather than remaining constant throughout the replenishment period. As such, researchers studying inventory management have considered different forms of demand behaviour, which include linear, quadratic, cubic and even higher order models. For example, Khanra and Chaudhuri (2003) analysed inventory systems characterized by quadratic time-varying demand and deterioration. More flexible higher order demand function allows one to model accelerated or decelerated behaviour and changes in the growth of the demand pattern. Recent literature suggests that developing flexible models of demand is among the current issues in inventory research.

Mohanty *et al*., (2024) analysed the two-phase demand function composed of constant and quadratic parts, whereas Pathak et al., (2024a) focused on deteriorating items with exponential demand, time varying holding cost and partial backlogging. Pathak *et al.*, (2024b) also examined the bi-quadratic form of demand function in two warehouses inventory systems under shortages and inflation. The use of higher-order demand functions has also been accompanied by greater interest in more complex deterioration models. Paul *et al.*, (2024) developed a model for deteriorating products with cubic deterioration and demand rate which depends on shelf-life, price, stock and time factors. In the same way, recent research continued to focus on Weibull deterioration along with various characteristics of inventory system. The study of Gupta *et al.*, (2025) considered a model on Weibull deterioration with time-dependent holding cost and different types of demand rates using the approximation based on Maclaurin series expansion technique. The other study of Sheela and Rathnakumari (2025) dealt with a model on Weibull deterioration with credit period and preservation facility. From the above developments, the modelling and solution techniques of deterioration and resulting inventory equations received due attention.

The two-parameter Weibull distribution proves quite effective in inventory modelling due to the ability of these parameters to accommodate different patterns of deterioration. The combination of Weibull deterioration along with higher order demand functions results in a broader scope of modelling of inventory systems where demand as well as deterioration follows a nonlinear pattern with respect to time. This approach has been gaining a lot of popularity in contemporary inventory due to its relevance in modelling products such as fresh produce, pharmaceutical products, chemicals, and many others which deteriorate depending on how long they are stored in warehouse, while at the same time their demand may fluctuate at different stages of their life cycle. Some recent examples are given by Ramya and Sivakumar (2026), who study EOQ model with quartic demand, Weibull deterioration and time-dependent partial backlogging. Imprecise nature of inventory parameters is another important factor of consideration in modelling the inventory problem.

In real inventory systems, the demand, deteriorating process, and other cost parameters could be uncertain. The introduction of fuzzy set has been used to model these uncertainties. Thus, fuzzy inventory models have been constructed for different types of demand and deteriorating processes. One of the most prominent contributions towards the modelling of fuzzy deteriorating inventories has been made by Sarkar and Chakrabarti (2012).

Recent work on fuzzy inventories has also introduced models that are based on price-dependent demands, varying deterioration rates, and shortages. For example, Shaikh and Gite (2024) proposed a fuzzy inventory model in which demand was based on selling price, deterioration rate was variable, and shortages were completely backlogged. The optimised inventory model with fuzzy parameter has also been discussed by Ray *et al*., (2025), whereas the analysis of fuzzy inventory model with non-instantaneous deteriorating inventory has been conducted by Prabha *et al.*, (2026). Singh *et al*., (2026), also studied an inventory system under Weibull deterioration, selling price-dependent demand, and a fuzzy environment.

Model development for higher order demands and Weibull deterioration has formed part of our recent research. Specifically, Onyenike *et al*., (2026) focused on the study of fuzzy EOQ models with two-parameter Weibull deteriorating goods in relation to time-dependent cubic demand without shortages. Later, Onyenike *et al*., (2026b) extended this to an inventory system for two-parameter Weibull deteriorating goods by considering quartic demand without shortages. This analysis depicts the mathematical approach used in this paper as it involves the use of quartic demand and two-parameter Weibull deterioration. This trend was further extended to accommodate fuzzy inventory parameters in a paper by Onyenike *et al.*, (2026a), who used the fourth-order Runge–Kutta method to compute the numerical solutions of the inventory equations for the lower and upper α-cuts. This study extends the investigation of the same inventory model using an integrating factor method, which transforms the governing equations to an integral form. Since the resulting integrals involving the exponential Weibull function do not generally have convenient closed-form solutions, a Taylor series expansion is used to

solve the integrals term by term. An adaptive convergence criterion is established to determine the number of terms needed in the series.

The objective of this paper is therefore to develop an integrating-factor series solution for the model that has been developed earlier by Onyenike et al. (2026a) on the fuzzy inventory model for two-parameter Weibull deteriorating items with time-dependent quartic demand. Specifically, the study derives the lower and upper α-cut values in a series form and further develops the adaptive convergence criterion to determine the necessary number of series terms, obtain the optimal solution and perform a sensitivity analysis to show the effect of variation in the model parameters on the optimal costs. The study further performs a comparative test between the series solutions and those obtained from the fourth-order Runge-Kutta method under the same model assumptions.

Model Assumptions and Notation

2.1. Assumptions

The inventory system is developed based on the following assumptions

i. The inventory system considers only single item.

ii. The demand rate is time dependent quartic function with fuzzy parameters, i.e., $\tilde{D}(t) = \tilde{a}_1 + \tilde{a}_2 t + \tilde{a}_3 t^2 + \tilde{a}_4 t^3 + \tilde{a}_5 t^4, \tilde{a}_1,\ \ \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5 > 0$.where $\tilde{a}_1$ is the initial demand, $\tilde{a}_2$ initial rate of change, $\tilde{a}_3$ rapid growth, $\tilde{a}_4$ maturation point $\tilde{a}_5$ decline.

iii. Lead time is zero

iv. Replenishment rate is instantaneous

v. The deterioration rate follows two-parameter Weibull distribution with fuzzy scale $\tilde{\gamma} > 0$ and shape parameter $\tilde{\beta} > 0$, i.e., $\tilde{\theta}(t) = \tilde{\gamma}\tilde{\beta}t^{\tilde{\beta}-1}$

vi. Fuzzy parameters are modelled as a triangular number,

$$\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3),\ \tilde{\beta} = (\beta_1, \beta_2, \beta_3),$$
$$\tilde{a}_i = (a_{i1}, a_{i2}, a_{i3}),\ i = 1, 2, 3, 4, 5$$

vii. No Shortages are allowed

2.2 Notations

The notations and its meaning as used in this paper is represented in Table 1.

Table 1: Notations and meanings as used in this work

Notation	Meaning
$I(t)$	Inventory level at time t
Q_0	Initial quantity
T	Inventory cycle length
$\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5$	Fuzzy demand parameters
$\tilde{D}(t)$	Fuzzy time dependent demand
$\tilde{\gamma}, \tilde{\beta}$	Weibull Scale and shape parameters
$I_L(t, \alpha)$	α-cut inventory lower bound
$I_U(t, \alpha)$	α-cut inventory upper bound

Notation	Meaning
$Q_{L0}(\alpha)$	α-cut initial quantity lower bound
$Q_{U0}(\alpha)$	α-cut initial quantity upper bound
$\tilde{\theta}_W(t)$	Fuzzy Weibull deterioration rate
$\widetilde{TC}_W$	Total fuzzy inventory cost
TC_W	Total inventory cost
AC_W	Average cost
$\widetilde{HC}$	Fuzzy holding cost per cycle
OC	Ordering cost
$\widetilde{PC}$	Purchasing cost per cycle
c_0	Ordering cost per unit
α	Membership level
c_h	Holding cost per unit
T^*	Optimal inventory cycle length
$\tilde{Q}_0$	Optimal quantity
$\widetilde{DC}$	Fuzzy deterioration cost
d_c	Deterioration cost per unit
p_c	Purchasing cost per unit

3. Mathematical Model Formulation

Let $I(t)$ be an inventory level at time t. We consider a fuzzy two-parameter Weibull distribution $\tilde{\theta}(t) = \tilde{\gamma}\tilde{\beta}t^{\tilde{\beta}-1}$ with quartic time dependent demand function $\widetilde{D}(t) = \tilde{a}_1 + \tilde{a}_2 t + \tilde{a}_3 t^2 + \tilde{a}_4 t^3 + \tilde{a}_5 t^4$earlier developed by Onyenike *et al.* (2026a) given by the equation

$$\frac{dI(t)}{dt} + \tilde{\gamma}\tilde{\beta}t^{\tilde{\beta}-1}I(t) = -(\tilde{a}_1 + \tilde{a}_2 t + \tilde{a}_3 t^2 + \tilde{a}_4 t^3 + \tilde{a}_5 t^4) \quad (1)$$

With boundary conditions $I(0) = Q_0$ *and* $I(T) = 0$

Where $\tilde{\gamma} > 0$ and $\tilde{\beta} > 0$ are scale and shape parameters $for\ Weibull\ distribution$, and $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5 > 0$ the demand parameters. Figure 1 below shows the positive inventory level at time t.

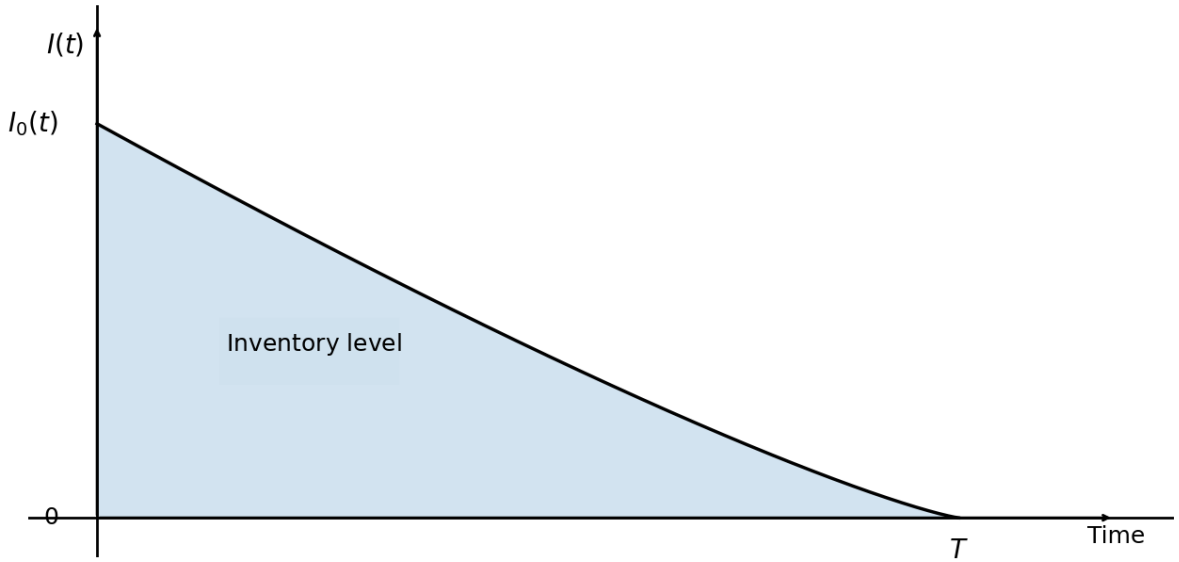


Figure. 1. Inventory Level dynamics over Time

To solve the differential equation (1), the decomposition into an upper and lower α-cut of equation (1) is given as:

$$\frac{dI_L(t,\alpha)}{dt} + \gamma_L \beta_L t^{\beta_L - 1} I_L(t,\alpha) = -(a_{1L}(\alpha) + a_{2L}(\alpha)t + a_{3L}(\alpha)t^2 + a_{4L}(\alpha)t^3 + a_{5L}(\alpha)t^4) \tag{2}$$

with boundary $I_L(0) = Q_{0L}$ and $I_L(T,\alpha) = 0$

and

$$\frac{dI_U(t,\alpha)}{dt} + \gamma_U \beta_U t^{\beta_U - 1} I_U(t,\alpha) = -(a_{1U}(\alpha) + a_{2U}(\alpha)t + a_{3U}(\alpha)t^2 + a_{4U}(\alpha)t^3 + a_{5U}(\alpha)t^4) \tag{3}$$

with boundary condition $I_U(0) = Q_{0U}$ and $I_U(T,\alpha) = 0$

Equation (2) and (3) above are linear differential equation. While solving them we find the integrating factor (I.F) as follows:

$$\mu_U(t,\alpha) = e^{\int \gamma_U \beta_U t^{\beta_U - 1} dt} = e^{\gamma_U t^{\beta_U}}$$

$$\mu_L(t,\alpha) = e^{\int \gamma_L \beta_L t^{\beta_L - 1} dt} = e^{\gamma_L t^{\beta_L}}$$

Applying that to equation (2) and (3) with the boundary conditions yields;

$$I_L(t,\alpha) = e^{-\gamma_L t^{\beta_L}} \int_t^T (a_{1L} + a_{2L}s + a_{3L}s^2 + a_{4L}s^3 + a_{5L}s^4) e^{\gamma_L s^{\beta_L}} ds \tag{4}$$

and

$$I_U(t,\alpha) = e^{-\gamma_U t^{\beta_U}} \int_t^T (a_{1U} + a_{2U}s + a_{3U}s^2 + a_{4U}s^3 + a_{5U}s^4) e^{\gamma_U s^{\beta_U}} ds \tag{5}$$

The integrating factor method applied yields an exact representation of the lower and upper α-cut inventory functions. Thus, there appear integrals of the form;

$$\int_t^T s^k e^{\gamma_U s^{\beta_U}} ds,\ k = 0,1,2,3,4.$$

Even though the integrating-factor approach provides an exact integral representation, the integral involving the exponential of the Weibull function does not have a convenient closed form for all possible values of the shape parameter. Thus, the Taylor series expansion of the exponential term is employed to provide an expandable solution series.

The solutions to equations (4) and (5) therefore are therefore given by

$$I_L(t,\alpha) = e^{-\gamma_L t^{\beta_L}} \sum_{n=0}^{\infty} \frac{\gamma_L^n}{n!}\left[\frac{a_{1L}(T^{n\beta_L+1}-t^{n\beta_L+1})}{n\beta_L+1} + \frac{a_{2L}(T^{n\beta_L+2}-t^{n\beta_L+2})}{n\beta_L+2} + \frac{a_{3L}(T^{n\beta_L+3}-t^{n\beta_L+3})}{n\beta_L+3} + \frac{a_{4L}(T^{n\beta_L+4}-t^{n\beta_L+4})}{n\beta_L+4} + \frac{a_{5L}(T^{n\beta_L+5}-t^{n\beta_L+5})}{n\beta_L+5}\right] \quad (6)$$

Applying the boundary condition $I_L(0) = Q_{0L}$ in equation (6) we have,

$$Q_{0L}(\alpha) = \sum_{n=0}^{\infty} \frac{\gamma_{L(\alpha)}^n}{n!}\left[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1} + \frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2} + \frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3} + \frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4} + \frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\right] \quad (7)$$

and

$$I_U(t,\alpha) = e^{-\gamma_U t^{\beta_U}} \sum_{n=0}^{\infty} \frac{\gamma_U^n}{n!}\left[\frac{a_{1U}(T^{n\beta_U+1}-t^{n\beta_U+1})}{n\beta_U+1} + \frac{a_{2U}(T^{n\beta_U+2}-t^{n\beta_U+2})}{n\beta_U+2} + \frac{a_{3U}(T^{n\beta_U+3}-t^{n\beta_U+3})}{n\beta_U+3} + \frac{a_{4U}(T^{n\beta_U+4}-t^{n\beta_U+4})}{n\beta_U+4} + \frac{a_{5U}(T^{n\beta_U+5}-t^{n\beta_U+5})}{n\beta_U+5}\right] \quad (8)$$

Also applying the boundary condition $I_U(0) = Q_{0U}$ in (8), we have:

$$Q_{0U}(\alpha) = \sum_{n=0}^{\infty} \frac{\gamma_{U(\alpha)}^n}{n!}\left[\frac{a_{1U}(\alpha)T^{n\beta_U(\alpha)+1}}{n\beta_U(\alpha)+1} + \frac{a_{2U}(\alpha)T^{n\beta_U(\alpha)+2}}{n\beta_U(\alpha)+2} + \frac{a_{3U}(\alpha)T^{n\beta_U(\alpha)+3}}{n\beta_U(\alpha)+3} + \frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4} + \frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U(\alpha)+5}\right] \quad (9)$$

with convergence condition

$$\frac{|S_N - S_{N-1}|}{max(1,|S_N|)} < \varepsilon \quad (10)$$

with $\varepsilon = 10^{-8}$.

In this case, S_N and S_{N-1} denote successive partial sums. The computation of the series stops once the relative weight of the latest term is less than the specified tolerance, which is $\varepsilon = 10^{-8}$. The number of terms needed is a function of the period T and the model parameters.

The various costs associated with the inventory are;

Total fuzzy inventory cost $(\widetilde{TC_W}(T)) = OC(T) + PC(T)_\alpha + [HC]_\alpha + [DC(T)]_\alpha$ (11)

with

Ordering cost $OC(T) = C_0$

Fuzzy purchasing cost $PC(T)_\alpha = [C_p Q_{0L}, C_p Q_{0U}]$=

$$\left[\left(\sum_{n=0}^{\infty}\frac{C_p\gamma_{L(\alpha)}^n}{n!}\left[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\right.\right.\right.$$

$$\left.\left.\left.\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\right]\right),\left(\sum_{n=0}^{\infty}\frac{C_p\gamma_{U(\alpha)}^n}{n!}\left[\frac{a_{1U}(\alpha)T^{n\beta_U(\alpha)+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U(\alpha)+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U(\alpha)+3}}{n\beta_U(\alpha)+3}+\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+\right.\right.$$

$$\left.\left.\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U(\alpha)+5}\right]\right)\right] \tag{12}$$

Fuzzy holding cost $[HC]_\alpha = [HC_L(T), HC_U(T)]$

with

$$HC_U(T) = C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_U^n(-\gamma_U)^m}{n!m!(m\beta_U+1)}\left[\frac{a_{1U}T^{(m+n)\beta_U+2}}{(m+n)\beta_U+2}+\frac{a_{2U}T^{(m+n)\beta_U+3}}{(m+n)\beta_U+3}+\frac{a_{3U}T^{(m+n)\beta_U+4}}{(m+n)\beta_U+4}+\frac{a_{4U}T^{(m+n)\beta_U+5}}{(m+n)\beta_U+5}+\right.$$

$$\left.\frac{a_{5U}T^{(m+n)\beta_U+6}}{(m+n)\beta_U+6}\right] \tag{13}$$

and

$$HC_L(T) = C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_L^n(-\gamma_L)^m}{n!m!(m\beta_L+1)}\left[\frac{a_{1L}T^{(m+n)\beta_L+2}}{(m+n)\beta_L+2}+\frac{a_{2L}T^{(m+n)\beta_L+3}}{(m+n)\beta_L+3}+\frac{a_{3L}T^{(m+n)\beta_L+4}}{(m+n)\beta_L+4}+\frac{a_{4L}T^{(m+n)\beta_L+5}}{(m+n)\beta_L+5}+\right.$$

$$\left.\frac{a_{5L}T^{(m+n)\beta_L+6}}{(m+n)\beta_L+6}\right] \tag{14}$$

Fuzzy deterioration cost $[DC]_\alpha = [DC_L(T), DC_U(T)]$

with

$$DC_L(T) = d_c\left(Q_{0L} - \int_0^T D_L(t)dt\right)$$

$$= d_c\left(Q_{0L} - \int_0^T (a_{1L}(\alpha) + a_{2L}(\alpha)t + a_{3L}(\alpha)t^2 + a_{4L}(\alpha)t^3 + a_{5L}(\alpha)t^4)dt\right)$$

$$= d_c\left(Q_{0L} - \left(a_{1L}T + \frac{a_{2L}T^2}{2} + \frac{a_{3L}T^3}{3} + \frac{a_{4L}T^4}{4} + \frac{a_{5L}T^5}{5}\right)\right)$$

$$DC_L(T) = d_c\left(\sum_{n=0}^{\infty}\frac{\gamma_{L(\alpha)}^n}{n!}\left[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\right]-\right.$$

$$\left.\left(a_{1L}T + \frac{a_{2L}T^2}{2} + \frac{a_{3L}T^3}{3} + \frac{a_{4L}T^4}{4} + \frac{a_{5L}T^5}{5}\right)\right) \tag{15}$$

Similarly,

$$DC_U(T) = d_c\left(\sum_{n=0}^{\infty}\frac{\gamma_{U(\alpha)}^n}{n!}\left[\frac{a_{1U}(\alpha)T^{n\beta_U+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U+3}}{n\beta_U(\alpha)+3}+\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+\right.\right.$$

$$\left.\left.\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U+5}\right]-\left(a_{1U}T + \frac{a_{2U}T^2}{2} + \frac{a_{3U}T^3}{3} + \frac{a_{4U}T^4}{4} + \frac{a_{5U}T^5}{5}\right)\right) \tag{16}$$

The total fuzzy inventory cost in equation (11) now becomes:

$$\widetilde{TC_W}(T) = C_0 + \left[\left(\sum_{n=0}^{\infty}\frac{C_p\gamma_{L(\alpha)}^n}{n!}\left[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\right.\right.\right.$$

$$\left.\left.\left.\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\right]\right),\left(\sum_{n=0}^{\infty}\frac{C_p\gamma_{U(\alpha)}^n}{n!}\left[\frac{a_{1U}(\alpha)T^{n\beta_U(\alpha)+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U(\alpha)+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U(\alpha)+3}}{n\beta_U(\alpha)+3}+\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+\right.\right.$$

$$\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U(\alpha)+5}\Big]\Big] + \Big[\Big(C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_L^n(-\gamma_L)^m}{n!m!(m\beta_L+1)}\Big[\frac{a_{1L}T^{(m+n)\beta_L+2}}{(m+n)\beta_L+2}+\frac{a_{2L}T^{(m+n)\beta_L+3}}{(n+m)\beta_L+3}+\frac{a_{3L}T^{(m+n)\beta_L+4}}{(n+m)\beta_L+4}+$$
$$\frac{a_{4L}T^{(m+n)\beta_L+5}}{(n+m)\beta_L+5}+\frac{a_{5L}T^{(m+n)\beta_L+6}}{(n+m)\beta_L+6}\Big]\Big),\Big(C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_U^n(-\gamma_U)^m}{n!m!(m\beta_U+1)}\Big[\frac{a_{1U}T^{(m+n)\beta_U+2}}{(m+n)\beta_U+2}+\frac{a_{2U}T^{(m+n)\beta_U+3}}{(n+m)\beta_U+3}+$$
$$\frac{a_{3U}T^{(m+n)\beta_U+4}}{(n+m)\beta_U+4}+\frac{a_{4U}T^{(m+n)\beta_U+5}}{(n+m)\beta_U+5}+\frac{a_{5U}T^{(m+n)\beta_U+6}}{(n+m)\beta_U+6}\Big]\Big)\Big] + \Big[\Big(d_c\Big(\sum_{n=0}^{\infty}\frac{\gamma_{L(\alpha)}^n}{n!}\Big[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+$$
$$\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\Big]-\Big(a_{1L}T+\frac{a_{2L}T^2}{2}+\frac{a_{3L}T^3}{3}+\frac{a_{4L}T^4}{4}+$$
$$\frac{a_{5L}T^5}{5}\Big)\Big)\Big),\Big(d_c\Big(\sum_{n=0}^{\infty}\frac{\gamma_{U(\alpha)}^n}{n!}\Big[\frac{a_{1U}(\alpha)T^{n\beta_U+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U+3}}{n\beta_U(\alpha)+3}+\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+$$
$$\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U+5}\Big]-\Big(a_{1U}T+\frac{a_{2U}T^2}{2}+\frac{a_{3U}T^3}{3}+\frac{a_{4U}T^4}{4}+\frac{a_{5U}T^5}{5}\Big)\Big)\Big)\Big] \quad (17)$$

We apply the graded mean integration representation (GMIR) to defuzzify the total fuzzy inventory cost in (14) into crisp and the resulting total inventory cost gives:

$$TC_W(T) = C_o + \frac{TC_{WL}(T)+4TC_{WM}(T)+TC_{WU}(T)}{6} \quad (18)$$

with

$$TC_{WL}(T) = \Big(\sum_{n=0}^{\infty}\frac{C_p\gamma_{L(\alpha)}^n}{n!}\Big[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\Big]\Big)+$$
$$C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_L^n(-\gamma_L)^m}{n!m!(m\beta_L+1)}\Big[\frac{a_{1L}T^{(m+n)\beta_L+2}}{(m+n)\beta_L+2}+\frac{a_{2L}T^{(m+n)\beta_L+3}}{(n+m)\beta_L+3}+\frac{a_{3L}T^{(m+n)\beta_L+4}}{(n+m)\beta_L+4}+\frac{a_{4L}T^{(m+n)\beta_L+5}}{(n+m)\beta_L+5}+\frac{a_{5L}T^{(m+n)\beta_L+6}}{(n+m)\beta_L+6}\Big]+$$
$$d_c\Big(\sum_{n=0}^{\infty}\frac{\gamma_{L(\alpha)}^n}{n!}\Big[\frac{a_{1L}(\alpha)T^{n\beta_L+1}}{n\beta_L(\alpha)+1}+\frac{a_{2L}(\alpha)T^{n\beta_L+2}}{n\beta_L(\alpha)+2}+\frac{a_{3L}(\alpha)T^{n\beta_L+3}}{n\beta_L(\alpha)+3}+\frac{a_{4L}(\alpha)T^{n\beta_L(\alpha)+4}}{n\beta_L(\alpha)+4}+\frac{a_{5L}(\alpha)T^{n\beta_L(\alpha)+5}}{n\beta_L+5}\Big]-\Big(a_{1L}T+$$
$$\frac{a_{2L}T^2}{2}+\frac{a_{3L}T^3}{3}+\frac{a_{4L}T^4}{4}+\frac{a_{5L}T^5}{5}\Big)\Big) \quad (19)$$

$$TC_{WM}(T) = \Big(\sum_{n=0}^{\infty}\frac{C_p\gamma_{M(\alpha)}^n}{n!}\Big[\frac{a_{1M}(\alpha)T^{n\beta_M+1}}{n\beta_M(\alpha)+1}+\frac{a_{2M}(\alpha)T^{n\beta_M+2}}{n\beta_M(\alpha)+2}+\frac{a_{3M}(\alpha)T^{n\beta_M+3}}{n\beta_M(\alpha)+3}+\frac{a_{4M}(\alpha)T^{n\beta_M(\alpha+4}}{n\beta_M(\alpha)+4}+$$
$$\frac{a_{5M}(\alpha)T^{n\beta_M(\alpha)+5}}{n\beta_M+5}\Big]\Big)+C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_M^n(-\gamma_M)^m}{n!m!(m\beta_M+1)}\Big[\frac{a_{1M}T^{(m+n)\beta_M+2}}{(m+n)\beta_M+2}+\frac{a_{2M}T^{(m+n)\beta_M+3}}{(n+m)\beta_M+3}+\frac{a_{3M}T^{(m+n)\beta_M+4}}{(n+m)\beta_M+4}+$$
$$\frac{a_{4M}T^{(m+n)\beta_M+5}}{(n+m)\beta_M+5}+\frac{a_{5M}T^{(m+n)\beta_M+6}}{(n+m)\beta_M+6}\Big]+d_c\Big(\sum_{n=0}^{\infty}\frac{\gamma_{M(\alpha)}^n}{n!}\Big[\frac{a_{1M}(\alpha)T^{n\beta_M+1}}{n\beta_M(\alpha)+1}+\frac{a_{2M}(\alpha)T^{n\beta_M+2}}{n\beta_M(\alpha)+2}+\frac{a_{3M}(\alpha)T^{n\beta_M+3}}{n\beta_M(\alpha)+3}+$$
$$\frac{a_{4M}(\alpha)T^{n\beta_M(\alpha)+4}}{n\beta_M(\alpha)+4}+\frac{a_{5M}(\alpha)T^{n\beta_M(\alpha)+5}}{n\beta_M+5}\Big]-\Big(a_{1M}T+\frac{a_{2M}T^2}{2}+\frac{a_{3M}T^3}{3}+\frac{a_{4M}T^4}{4}+\frac{a_{5M}T^5}{5}\Big)\Big) \quad (20)$$

$$TC_{WU}(T) = \Big(\sum_{n=0}^{\infty}\frac{C_p\gamma_{U(\alpha)}^n}{n!}\Big[\frac{a_{1U}(\alpha)T^{n\beta_U(\alpha)+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U(\alpha)+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U(\alpha)+3}}{n\beta_U(\alpha)+3}+\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+$$
$$\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U(\alpha)+5}\Big]\Big)+C_h\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_U^n(-\gamma_U)^m}{n!m!(m\beta_U+1)}\Big[\frac{a_{1U}T^{(m+n)\beta_U+2}}{(m+n)\beta_U+2}+\frac{a_{2U}T^{(m+n)\beta_U+3}}{(n+m)\beta_U+3}+\frac{a_{3U}T^{(m+n)\beta_U+4}}{(n+m)\beta_U+4}+$$
$$\frac{a_{4U}T^{(m+n)\beta_U+5}}{(n+m)\beta_U+5}+\frac{a_{5U}T^{(m+n)\beta_U+6}}{(n+m)\beta_U+6}\Big]+d_c\Big(\sum_{n=0}^{\infty}\frac{\gamma_{U(\alpha)}^n}{n!}\Big[\frac{a_{1U}(\alpha)T^{n\beta_U+1}}{n\beta_U(\alpha)+1}+\frac{a_{2U}(\alpha)T^{n\beta_U+2}}{n\beta_U(\alpha)+2}+\frac{a_{3U}(\alpha)T^{n\beta_U+3}}{n\beta_U(\alpha)+3}+$$
$$\frac{a_{4U}(\alpha)T^{n\beta_U(\alpha)+4}}{n\beta_U(\alpha)+4}+\frac{a_{5U}(\alpha)T^{n\beta_U(\alpha)+5}}{n\beta_U+5}\Big]-\Big(a_{1U}T+\frac{a_{2U}T^2}{2}+\frac{a_{3U}T^3}{3}+\frac{a_{4U}T^4}{4}+\frac{a_{5U}T^5}{5}\Big)\Big) \quad (21)$$

And the average cost is given as:

$$AC_w(T) = \frac{TC_w(T)}{T} \quad (22)$$

and optimal Average cost $AC_w(T^*) = \min_{T>0} AC_w(T)$ (23)

4. Numerical Example

To demonstrate the effectiveness of our developed model, let us consider the following parameters;
$a_1 = (14, 15, 16)$ $a_2 = (5,6, 7)$ $a_3 = (7,8,9)$ $a_4 = (3,4,5)$ $a_5 = (1,2,3)$ $\gamma = (0.1,0.2,0.3)$ $\beta = (1,2,3)$, $d_c = 6$, $C_h = 3$, $C_o = 90$, $C_p = 8$ $T = 2$

Using the parameter values above and applying the total inventory cost equation, we find the total inventory Cost, average cost and optimal T^* by Python application software as: Total inventory cost $TC_W(T) = 2744.654343$, average cost $AC_w(T) = 1372.327171, TC_W(T^*) = 229.581559$, optimal cycle length $T^* = 0.738749$, $Q^* = 15$

and optimal average cost $AC_w(T^*) = 310.770661$

To examine the convergence behaviour of the Taylor series a numerical analysis was conducted using certain cycle times T = (0.5, 0.738749, 1.0, 1.5, and 2.0) along with the lowest, middle, and highest values of the fuzzy parameter. The tolerance value used for convergence was $\varepsilon = 10^{-8}$. The number of terms for convergence in both single and double Taylor series approaches are presented in Table 2.

Table 2: Convergence Taylor-Series Expansions at Selected Cycle Lengths.

Cycle length, *T*	Single series, *N (L, M, U)*	Double series, *K (L, M, U)*
0.500000	(6, 6, 5)	(6, 6, 5)
0.738749	(6, 7, 7)	(6, 6, 7)
1.000000	(7, 7, 8)	(6, 7, 8)
1.500000	(7, 9, 12)	(7, 9, 11)
2.000000	(8, 11, 16)	(7, 11, 16)

As seen from the above table, there is a tendency for an increase in the number of terms as the cycle length increases, although the degree at which this happens varies with the fuzzy parameter level. It can be observed that the series expansion demands an average of 8 terms at each level of fuzziness.

5. Sensitivity Analysis

In order to study effects of changes on demand, Weibull parameters and the associated inventory cost on the total cost, average cost, optimal cost, optimal cycle length and optimal quantity, a sensitivity analysis is conducted by varying the parameters by $\pm 20\%$ taking one parameter at a time and keeping the remaining parameters constant. The results are shown in **Table 3** and **Table 4**. **Table 4** shows the result for the Fuzzy model. **Table 3** is the result for the optimal cost and optimal cycle length

Table 3: Sensitivity Analysis of Total and Average Inventory Costs to ±20% Changes in Model Parameters.

Par	Baseline	Δ% Par	Δ in Baseline	$TC_W(T)$	Δ% $TC_W(T)$	$AC_w(T)$	Δ% $AC_w(T)$
a_1	15	−20	12	2002.112506	−4.802887	1001.056253	−4.802887
a_1	15	+20	18	2204.133778	+4.802887	1102.066889	+4.802887
a_2	6	−20	4.8	2052.530767	−2.405583	1026.265383	−2.405583
a_2	6	+20	7.2	2153.715518	+2.405583	1076.857759	+2.405583
a_3	8	−20	6.4	2002.090989	−4.803911	1001.045494	−4.803911
a_3	8	+20	9.6	2204.155296	+4.803911	1102.077648	+4.803911
a_4	4	−20	3.2	2021.651393	−3.873846	1010.825696	−3.873846
a_4	4	+20	4.8	2184.594892	+3.873846	1092.297446	+3.873846
a_5	2	−20	1.6	2034.605429	−3.257903	1017.302714	−3.257903
a_5	2	+20	2.4	2171.640856	+3.257903	1085.820428	+3.257903
γ	0.2	−20	0.16	1879.841990	−10.616647	939.920995	−10.616647
γ	0.2	+20	0.24	2353.827886	+11.920593	1176.913943	+11.920593
β	2	−20	1.6	1888.108807	−10.223573	944.054403	−10.223573
β	2	+20	2.4	2421.281052	+15.127878	1210.640526	+15.127878
d_c	6	−20	4.8	2040.064098	−2.998352	1020.032049	−2.998352
d_c	6	+20	7.2	2166.182186	+2.998352	1083.091093	+2.998352
C_h	3	−20	2.4	1995.049616	−5.138716	997.524808	−5.138716
C_h	3	+20	3.6	2211.196668	+5.138716	1105.598334	+5.138716
C_o	90	−20	72	2085.123142	−0.855870	1042.561571	−0.855870
C_o	90	+20	108	2121.123142	+0.855870	1060.561571	+0.855870
C_p	8	−20	6.4	1871.631084	−11.007062	935.815542	−11.007062
C_p	8	+20	9.6	2334.615201	+11.007062	1167.307600	+11.007062

Table 4. Sensitivity Analysis of Optimal Costs to ±20% Changes in Model Parameters.

Para	Δ%in Par	T^*	Q^*	$TC_W(T^*)$	$AC_w(T^*)$	$\%\Delta TC_W(T^*)$	$\%\Delta AC_w(T^*)$
a_1	-20	0.751481	12.804532	211.718637	281.735116	-7.780643	-9.343078
a_1	+20	0.726569	16.680090	246.808666	339.690690	7.503698	9.305907
a_2	-20	0.749923	14.742019	229.594462	306.157541	0.005620	-1.484413
a_2	+20	0.727991	14.798740	229.531500	315.294247	-0.021805	1.455602
a_3	-20	0.752627	14.935137	231.507521	307.599429	0.838901	-1.020441
a_3	+20	0.725925	14.622042	227.802203	313.809411	-0.775043	0.977811
a_4	-20	0.744299	14.867976	230.644271	309.881355	0.462891	-0.286162
a_4	+20	0.733452	14.682200	228.571494	311.638076	-0.439959	0.279117
a_5	-20	0.740895	14.815678	230.051920	310.505418	0.204877	-0.085350
a_5	+20	0.736654	14.731022	229.123382	311.032555	-0.199571	0.084272
γ	-20	0.751985	15.028947	231.773356	308.215328	0.954692	-0.822257
γ	+20	0.726518	14.538128	227.573955	313.239268	-0.874462	0.794350
β	-20	0.733538	14.791811	230.442612	314.152319	0.375053	1.088152
β	+20	0.744067	14.800748	229.373394	308.269822	-0.090672	-0.804722
d_c	-20	0.743436	14.908791	230.315538	309.798834	0.319703	-0.312715
d_c	+20	0.734221	14.642260	228.876384	311.726685	-0.307157	0.307630
C_h	-20	0.752944	15.187381	230.302184	305.868855	0.313886	-1.577307
C_h	+20	0.725461	14.391956	228.906306	315.532035	-0.294123	1.532118
C_o	-20	0.690230	13.414395	197.125974	285.594724	-14.136843	-8.101131
C_o	+20	0.779686	15.990931	260.780410	334.468564	13.589441	7.625528
C_p	-20	0.766541	15.592175	213.488263	278.508755	-7.009838	-10.381259
C_p	+20	0.714380	14.079517	244.695644	342.528876	6.583318	10.219180

The sensitivity analysis provides useful lessons for management on the nature of the suggested inventory model. The result shows that changes in the value of model parameters affect the values of optimal cycle time, total inventory costs, and average inventory costs differently. This means that higher priority must be given to those parameters which are relatively more sensitive.

(i) Demand parameters:

Among the demand parameters, the initial demand coefficient a_1 has the largest effect on inventory costs, whereas the declining-stage demand coefficient a_5 has the smallest effect. This points out to the importance of making an accurate assessment of the initial demand value, which helps make the right replenishment decision. In terms of management, more efforts should be made to conduct market research, forecasting, and control of the initial demand for products. Accurate values of this parameter allow managers to make decisions about replenishment amounts and intervals, thus avoiding stock outs and excessive inventory. Even though other demand parameters have smaller effects on inventory costs, they can still be useful for describing variations in demand throughout the product life cycle. In addition, the elasticity curves shown in Figures 2–5 reveal the fact that the effect of demand parameters depends on the replenishment cycle length. Indeed, a_1 affects demand throughout the whole cycle, while other coefficients affect demand at various stages of the product life cycle.

(ii) Deterioration parameters

From the results, the scale parameter γ has a more significant impact on the inventory than the shape parameter β. Because the scale parameter γ controls the extent of deterioration, the estimation process is of great significance, especially for inventory goods that experience deterioration. The same applies to cases where $\beta > 1$, since the rate of deterioration increases with time. All the above results suggest that the managers must consider the deterioration properties of inventory goods in setting their replenishment period.

(iii) Costs parameters

Regarding the cost parameters, the ordering cost C_o has the most influence on total inventory cost, and that the purchasing cost C_p has considerable influence on average inventory cost. The findings above mean that a company's management will have to focus on cutting down ordering-related cost by using appropriate procurement system and effective supplier negotiation, among other things. Holding cost C_h influences on a moderate extent and calls for balancing between the frequencies of orders and holding cost. On the other hand, the deterioration cost d_c influences on a lesser extent.

Additionally, from the elasticities it can be observed that the effect of the cost parameters is dependent on the replenishment period because ordering, purchasing, holding, and deterioration costs react in different ways to replenishment policy. As such, managers will have to factor in all the cost elements when deciding on their policy.

In general, the sensitivity analysis highlights the initial demand coefficient a_1 and the ordering cost C_o as especially important factors in the inventory model under consideration. The purchase cost C_p is another factor that requires special consideration. Hence, it is advisable to focus forecasting efforts on estimating the initial demand, while operations need to be concentrated on controlling ordering and purchasing costs. The rest of the factors have relatively low impact, but it is still necessary to monitor them.

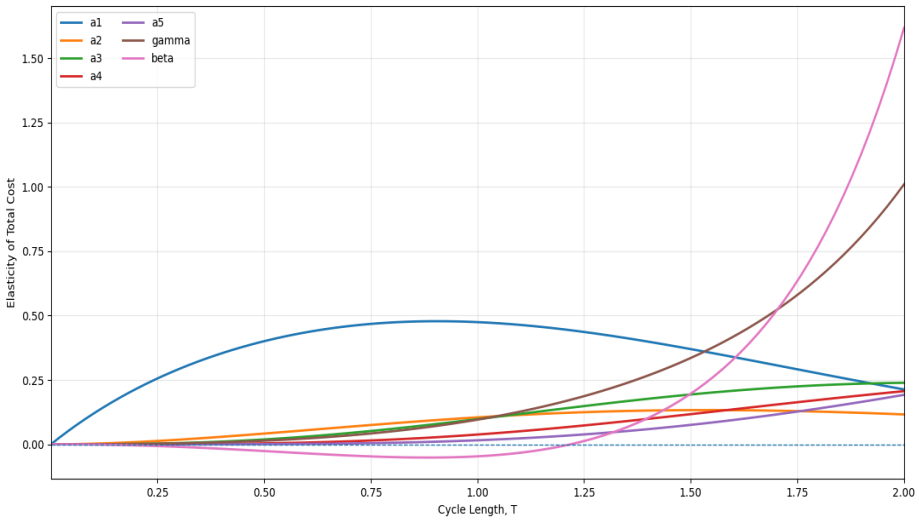


Figure 2: Elasticity of Total Cost with Respect to Demand and Weibull Deterioration Parameters against Cycle Length

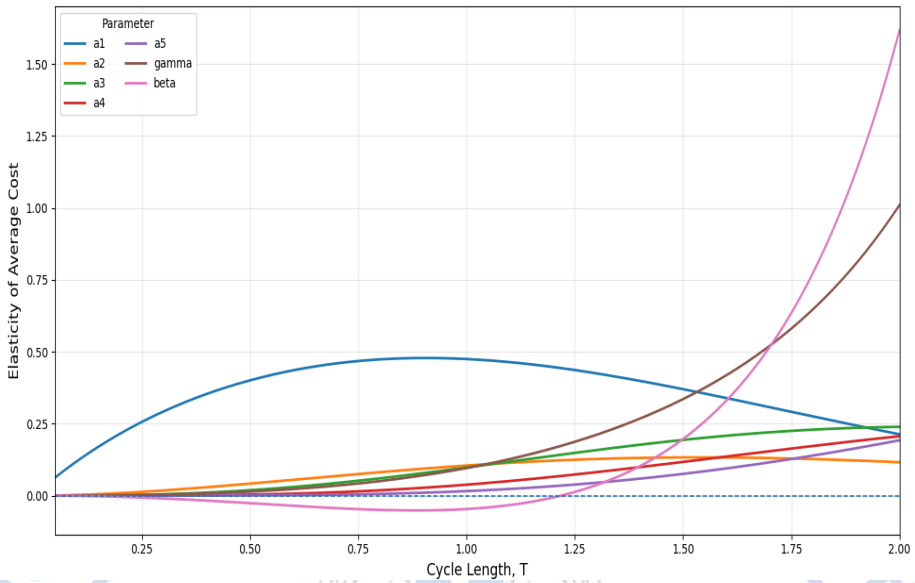


Figure 3*:* Elasticity of Average Cost with Respect to Demand and Weibull Deterioration Parameters against Cycle Length

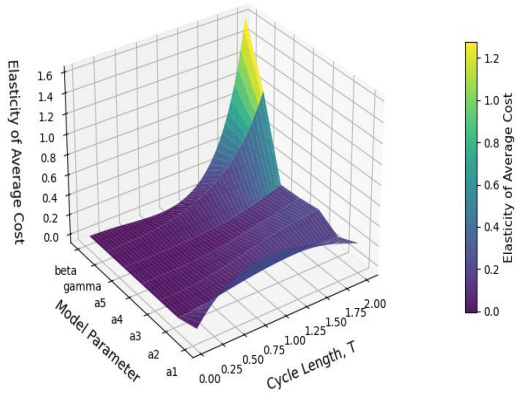


Figure 4: Three-dimensional representation of the elasticity of average cost with respect to demand and Weibull deterioration parameters against cycle length

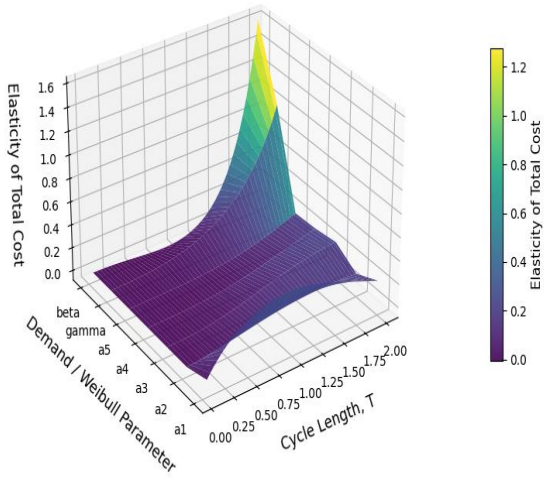


Figure 5: Three-dimensional representation of the elasticity of total cost with respect to demand and Weibull deterioration parameters against cycle length

Table 5: Comparative Analysis between RK4 result and integrating base series method result

Measure	RK4 at $T = 2$	Series at $T = 2$	RK4 at optimal T^*	Analytical Cost at optimal T^*
	2.000000	2.000000	0.738749	0.738749
Order quantity, Q_W	181	181	15	15
Purchasing cost, PC_W	1449.078359	1449.078359	118.182469	118.182470
Holding cost, HC_W	671.567047	671.567214	17.839055	17.839057
Deterioration cost, DC_W	534.008769	534.008769	3.560029	3.560029
Total cost, TC_W	2744.654176	2744.654343	229.581553	229.581556
Average cost, AC_W	1372.327088	1372.327171	310.770658	310.770661

Table 5 exhibits considerable agreement between RK4 method and integrating factor series expansion at the two analysis points. The values of the fuzzy total costs at T = 2 and at the optimal cycle length T^*= 0.738749 are 2744.654176, 2744.654343 and 229.581553, 229.581556 respectively, while that of their optimal average costs is 310.770658 and 310.770661 respectively.

When compared with the optimal cycle length value, the fuzzy order quantity at T = 2 reduces from 181 to 15, holding cost reduces from approximately 671.57 to 17.84 and the deterioration cost is reduced from 534.01 to 3.56. All these indicate the significant effect of cycle length on the requirements of inventory and their respective cost components.

$$E_{abs} = |C_s - C_{RK4}|$$

$$E_{abs}(\%) = \frac{|C_s - C_{RK4}|}{C_{RK4}} \times 100\%$$

Table 6. Absolute and Relative Differences between RK4 and Taylor-Series Results

Measure	**Absolute difference at T = 2**	**Relative difference at T = 2 (%)**	**Absolute difference at T* = 0.738749**	**Relative difference at T* = 0.738749 (%)**
Order quantity, Q_W	0.000000000	0.000000000	0.000000000	0.000000000
Purchasing cost, PC_W	0.000000000	0.000000000	0.000001000	0.000000846
Holding cost, HC_W	0.000167000	0.000024867	0.000002000	0.000011211
Deterioration cost, DC_W	0.000000000	0.000000000	0.000000000	0.000000000
Total cost, TC_W	0.000167000	0.000006085	0.000003000	0.000001307
Average cost, AC_W	0.000083000	0.000006048	0.000003000	0.000000965

As seen from the results given in Table 5, the solutions obtained by the two procedures are similar in ordered quantity at both points. The relative difference in total cost and average cost of inventory at $T = 2$ is 0.000006085% and 0.000006048%, correspondingly. At the optimal cycle period $T^* = 0.738749$ the respective relative differences are 0.000001307% and 0.000000965%, correspondingly. This behavior is been represented in Figure 6 below. The relative difference in purchase cost, holding cost, and deterioration cost are small as well. This shows a numerical agreement for both methodologies under same assumptions.

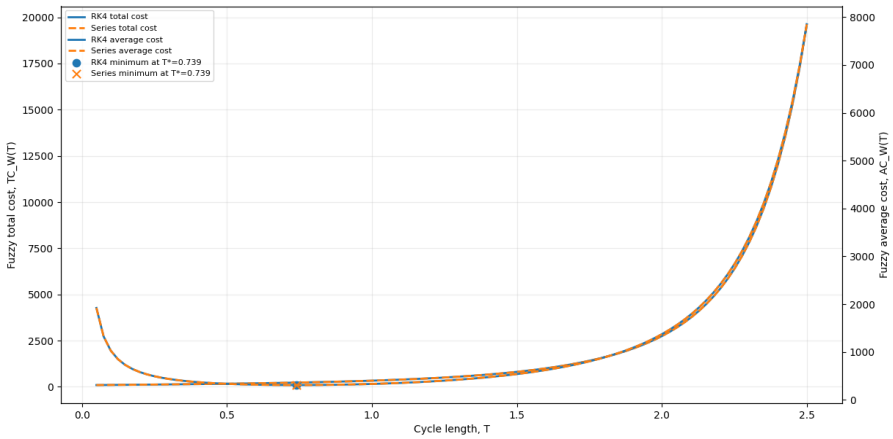


Figure 6. Fuzzy Total and Average Cost Curves: RK4 versus Integrating-Factor method

6. CONCLUSION

In this paper, we derived a model for deteriorating items with deterioration rate which follows Two-parameter Weibull distribution function and demand rate modelled as a time-dependent Quartic function. The Weibull parameters γ, β and demand parameters a_1,a_2, a_3, a_4 a_5 are considered fuzzy. The arameters associated with both deteriorating and demand rates have been expressed in terms of triangular fuzzy numbers. To deal with uncertainty and obtain a crisp inventory cost, the α-cut approach along with Graded Mean Integration Representation (GMIR) has been applied. This results in an optimal inventory policy subject to the no-shortage condition and includes finding of inventory position, optimal order quantity, total cost of inventory, average cost of inventory, and the optimal inventory cycle time.

The most significant advantage of the presented model framework is the ability to account for time varying demand, time dependent deteriorating process, and uncertainty in the parameters simultaneously. The quartic demand rate allows for modelling of complex demand patterns whereas Weibull distribution has provided a flexible deterioration process.

The sensitivity analysis demonstrates how variations in fuzzy Weibull parameters and fuzzy quartic demand coefficients influence the total cost and average inventory cost components. The results highlight the relative importance of fuzziness in determining optimal system performance. However, there are some

limitations in this paper. The model considers the case of a single item, instantaneous ordering, zero lead time, and no shortage. Also, it uses quartic demand function along with triangular fuzzy numbers which may fail to represent all types of demands and uncertainties that can occur. Nevertheless, the model will work in industries of perishable items like food, drugs, agriculture, chemicals, etc. With no-shortage policy, this study restricts its analysis on a single time of positive inventories in the ordering cycle. The presence of shortages will result in the division of the entire cycle into positive-inventory periods and shortage periods, which will be based on separate differential equations, boundary conditions, and the time range of the integration. Hence, this will require changes in the integrating factor solutions along with the Taylor series expansions and additionally will include shortage cost terms as well.

Future scope of this research includes considering the cases of shortages, non-zero lead times, price dependent demand, effect of inflation, delay in payments, different types of fuzzy numbers, and multi-items inventories.

Author Contributions

Conceptualization, K.O., J.T. and P.E.E.; methodology, K.O.; software, K.O.; validation, K.O., J.T. and P.E.E.; formal analysis, K.O.; investigation, K.O.; resources, K.O., J.T. and P.E.E.; data curation, K.O.; writing—original draft preparation, K.O.; writing—review and editing, K.O., J.T. and P.E.E.; visualization, K.O.; supervision, J.T. and P.E.E.; project administration, J.T. and P.E.E. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

The data used to support the findings of this study are included within the article. Additional data are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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